Print copy. This is the print copy of an article whose version of record is a web page. That page, with the data and code behind every figure, is at
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SYSTEM DESCRIPTION

Six Control Laws for Multi-Shaker Random Vibration: What Each Does and How It Is Built

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Evidence	**Simulator.** Hertzforge's code at main commit b587d5a1, read for every law, the shared closed loop and the protections; the published sources, each read to the extent stated; and a numerical check of the article's own equations against the shipped laws on small made-up structures. No hardware, and no result about how any law scores.
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Abstract

A multi-shaker random vibration test drives several shakers at once so that a set of accelerometers on the test article follow target spectra and, for every pair of them, a target coherence and phase. The rule that turns the targets and a measurement of the system into shaker drives is the control law. Hertzforge, an open-source structural dynamics suite, ships six: the pseudo-inverse with a closed-loop correction of a factor of the target (the default), the same pseudo-inverse without feedback, a shape-constrained law that drives only the directions the system responds to most, a law that takes its cross terms from the article's own response to uncorrelated drive, a law that chooses the cross terms needing the least drive, and Brandon Zwink's band-average law, which matches averages over fractional-octave bands rather than single frequency lines. This article explains in plain terms what a multi-shaker controller does, and for each law states what it does, the problem it solves as an equation, the derivation, the algorithm step by step with the file and function that does each step, every setting with its default and range, how it handles the edge cases, what it gives up by design, and where the build differs from its published source. It describes the protections that can stop a run whatever the law asks for, and how a reader can plug in a law of their own. Each law's equations, as written here, were implemented again without Hertzforge and compared with the shipped law on small made-up structures: the largest difference is 5e-10 dB in the pseudo-inverse laws' errors and 2e-7 relative in the minimum-drive law's solve, except in the closed loop after 22 loops, where a change of one part in 10^15 moves single lines as far. The article reports no score. A companion article reports how each law scores on the Open Benchmark Set.

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Summary

A random vibration test on one shaker has one control signal and one target spectrum. The control signal may be one accelerometer or a combination of several, an average for example; the controller raises or lowers the drive at each frequency until that signal reads its target. With several shakers on one article the controller has more to decide. Each shaker's drive reaches every control accelerometer, each control accelerometer has a target spectrum of its own, and the target says not only how strong the vibration is at each of them but how the motion at one point is related to the motion at another. The rule that decides the drives is the control law.

Hertzforge ships six control laws for this kind of test. Five come from the published literature and are Kevin Cross's implementations; the sixth, the band-average law, is Brandon Zwink's. They differ in what they ask the shakers to do, how they use the measured response, and what they give up when the target cannot be met exactly.

This article describes each law as it is built, to the standard of a journal article: the problem it solves, its derivation, its algorithm in the order the code runs it, its settings, its edge cases, its design intent, and its published source with each place the build departs from it. Section 1 explains the control problem without equations beyond the one that defines it. Section 2 describes the closed loop four of the laws share, then each law. Section 3 describes the protections the controller applies to every law. Section 4 says how to plug in a new law. Section 5 reports how the descriptions were checked against the code, including a numerical check. Section 6 says what the article does not claim and the known limits of the build. Appendix A gives the derivations in full, and Appendix B carries one frequency line of a made-up structure through every law, with numbers a reader can reproduce from the article's code.

Nothing here says how well a law does. That is the subject of the companion study (Cross 2026b), which cites this article for the laws.

1. The Control Problem in Plain Words

1.1 From One Shaker to Several

A random vibration target is a **spectrum**: how much vibration there is in each narrow slice of frequency, in units such as g^2/Hz. The controller works on a grid of such slices, called **lines**, typically a fraction of a hertz to a few hertz apart. A range of lines grouped together, for example a one-third octave, is a **band**. A test on one shaker has one control signal, which may be one accelerometer or a combination of several (an average, for example), and one target spectrum for it.

With several shakers, each control accelerometer has a target spectrum of its own, and a spectrum per accelerometer is not the whole story. Two accelerometers can each read their target spectrum while moving together, in opposition, or independently. Two more numbers per pair and per line describe that relation. The **coherence**, from 0 to 1, says how much of the motion at one point is linearly tied to the motion at the other. The **phase** says whether, at that frequency, they move together (0 degrees), in opposition (180 degrees), or somewhere between. The spectrum of one accelerometer is its **autospectrum**; the quantity that carries the coherence and phase of a pair is their **cross spectrum**.

All of these, for every accelerometer and every pair, form one square table per line: the **spectral density matrix**. Its diagonal holds the autospectra and its off-diagonal entries the cross spectra. The same holds for the shaker drives: the drive is not one voltage spectrum per shaker but a matrix per line, whose off-diagonal entries say how correlated each pair of drives is. A spectral density matrix of real signals can never ask for negative power in any combination of its channels. A matrix with that property is called **positive semi-definite**, and a drive matrix without it cannot be played.

Four terms from the first article in this journal (Cross and Zwink 2026) are used throughout. The **article** is the test article, the hardware under test. A **gauge** is an accelerometer on it. The gauges the controller reads and brings to their targets are the **control gauges**. A **held-out gauge** is recorded but never shown to the controller; it is used only to judge the test. A **system** is the article on its suspension or fixture with a given set of shakers and control gauges, as the controller sees it (the first article of this series calls it the rig). Systems fall into three classes: as many shakers as control gauges (a **square system**), more control gauges than shakers, and fewer. A system is written shakers by control gauges: a 6 x 12 system has six shakers and twelve control gauges.

What the controller knows of the system is the **measured frequency response**: on each line, a matrix H with one row per control gauge and one column per shaker, giving the acceleration at each gauge per volt of drive at each shaker (g/V), measured at the start of the run. If the drive matrix is G_{uu} (V²/Hz), the response matrix at the control gauges (g²/Hz) is

$$G_{yy} = H\, G_{uu}\, H^{H} \tag{1}$$

where H^{H} is the conjugate transpose of H (Appendix A.1). A multi-shaker controller's task is to choose G_{uu} on every line so that G_{yy} matches the target, G_{ref}, as closely as the system allows. Everything that follows is a way of doing that. The convention $G = E[XX^{H}]$ for a spectral density matrix is the one Smallwood uses for multi-shaker control (Smallwood 1999); as he notes, it is the complex conjugate of the cross spectrum as Bendat and Piersol define it. The published methods this article draws on are mapped in a survey in this journal (Cross 2026a), which this article cites for the prior work instead of repeating it.

1.2 A Run From Start to Finish

Figure 1 shows the order of a run at the software's defaults. The settings named here are in `config.rs` of `crates/hertzforge-mimo-random` (`Identification`, `Gates`, `Caps`, `Estimation`, `Protections`), and the order is `MimoRandomService::run_auto` in `service.rs`.

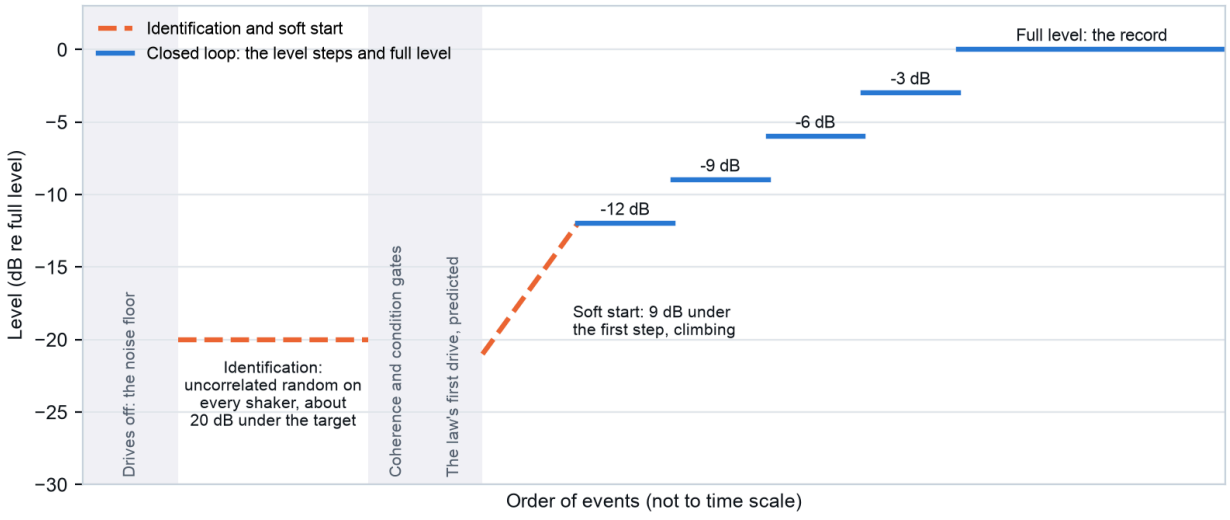


Figure 1: The order of a multiple-shaker random run in Hertzforge at its defaults: the noise floor, the identification, the gates, the predicted first drive, the soft start, the level steps and full level.

1. **The noise floor.** With the drives off, the controller records what the gauges read on their own (32 averaged frames by default).
2. **Identification.** Every shaker is driven at once with random noise that is uncorrelated from shaker to shaker. The level starts at 0.01 V RMS and is stepped, at most eight times, until the control gauges sit within 1 dB of

20 dB under the target (RMS over the band, averaged over the gauges). From the drives and the responses, averaged over ten frames per shaker and at least 64, the controller computes the measured frequency response H on every line. It also computes each control gauge's **multiple coherence**, the share of the gauge's response that the drives explain (1 when nothing but the drives moves it), and each line's **condition number**, defined in Section 1.4.

3. **The gates.** By default every control gauge's multiple coherence, averaged over the band, must reach 0.5, and every line's condition number must stay under 1,000. A run that fails a gate waits for the operator.
4. **The prediction.** The control law computes its first drive from the identification, and the controller predicts what it will do before anything is played. A first drive over a shaker's voltage ceiling waits for the operator.
5. **The soft start and the level steps.** The loop starts 9 dB under the first level step and climbs to it over 5 s, then steps up, by default from −12 dB to full level in 3 dB steps held 3 s each.
6. **Full level.** The loop runs at the target level until the test time is up or the operator stops it.

1.3 One Loop

From the soft start to the end, the controller repeats one loop (Figure 2), `loop_tick` in `service.rs`. It plays the drive in force, estimates the response matrix over the loop from 64 averaged frames (Hann window, half overlap), blends it into a running estimate with a memory of about four loops (a plain mean over the first two loops, then an exponential average that gives the newest loop a weight of 0.4), and refers it to full level: a response measured at −6 dB is scaled up by 6 dB before anything is judged or handed on. Every matrix the control law sees, and every drive it returns, is at full level. The controller owns the level, and a law cannot raise it.

Two kinds of loop are not handed to the law. The first loop after the loop engages is a settling loop: its estimate is discarded, and neither the abort lines nor the growing-error check of Section 3 judge it. A loop during which the level moved (the soft start's climb) is not judged against the abort lines or for a growing error, and is not handed to the law; the drive in force is played at the moving level. The drive-voltage, kinematic and limit-channel aborts apply in every loop. A level step of the schedule is made between loops and does not count as a moving level.

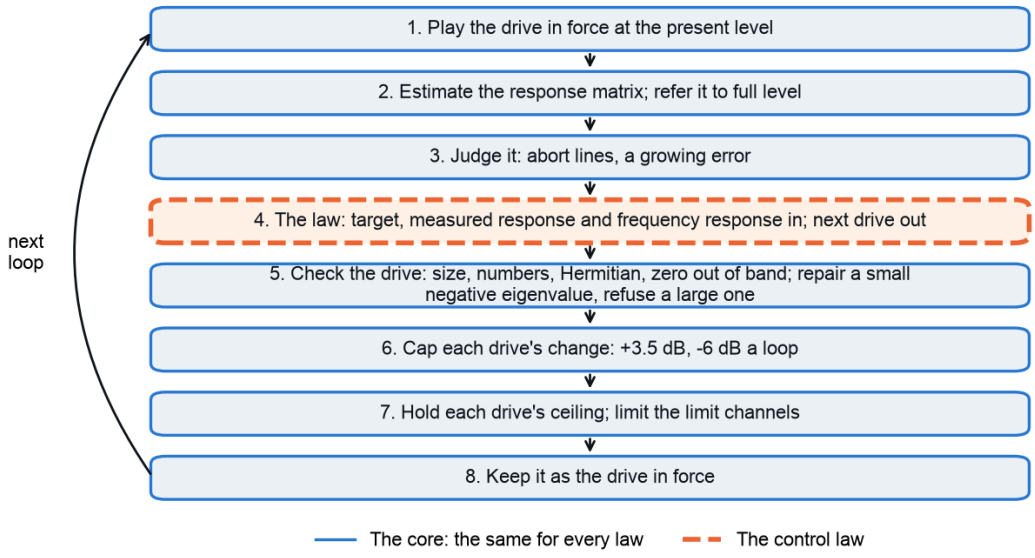


Figure 2: One loop of the run. The control law is one step; every other step belongs to the controller and is the same whatever law is running.

On each call the law is given the target with its alarm and abort lines, the measured response matrix, the measured frequency response, each gauge's multiple coherence, the estimate's degrees of freedom, the line spacing, the band and the loop count. A law may also ask for the last drive played, the loop's cross spectrum between response and drive, the identification's response or drive, or the noise floor. It returns the next drive matrix and whatever diagnostics it declared. The interface is `MimoControlLaw` in `crates/hertzforge-sdm` (`law` module): `seed` for the first drive, `iterate` for every one after.

The measured frequency response the law is given is, by default, the one from identification: measured once before the run and kept. The test may instead have it measured again during the run: every loop the controller forms a new estimate from the played drive and the measured response and blends it with the identification's by a weight the test sets (`FrfMode::Update` in `config.rs`).

1.4 What a System Can Reach

With as many shakers as control gauges and an exact frequency response, Equation 1 can be inverted line by line, and any target can in principle be met. With more control gauges than shakers it cannot: four shakers cannot set the spectra and cross spectra of twelve gauges independently. With fewer control gauges than shakers there are many drives that meet the target, and one has to be chosen.

The tool for all three cases is the **pseudo-inverse** H^{+} of the frequency response (V/g). The drive

$$G_{uu} = H^{+}\, G_{ref}\, H^{+H} \tag{2}$$

gives, with more gauges than shakers, the response closest to the target in the least-squares sense, which is the target projected onto what the system can produce, $PG_{ref}P^{H}$ with $P = HH^{+}$; with fewer gauges than shakers it gives, of all drives that meet the target exactly, the one of least total power (both derived in Appendix A.2). Smallwood uses the Moore-Penrose pseudo-inverse for both cases (Smallwood 1999). The distance between the target and what the system can reach is the **declared deviation** of the first article (Cross and Zwink 2026).

The pseudo-inverse is computed from the **singular value decomposition** of H on each line, $H = USV^{H}$. In plain terms, the columns of V are the patterns of shaker drive the system responds to, ordered from strongest to weakest, the diagonal of S (the singular values $s_1 \geq s_2 \geq \ldots$, g/V) says how strongly it responds to each, and the columns of U are the responses those patterns produce at the gauges. The ratio s_1/s_{min} is the line's **condition number**. A large one means some drive pattern barely moves the gauges, so meeting a target in that direction takes a large drive, and a small error in H becomes a large error in the drive. **Regularization** limits that: it drops or damps the weakest directions before inverting. With filter factors f_i (V/g), the regularized pseudo-inverse is

$$H^{+} = V\ \mathrm{diag}(f)\, U^{H}, \qquad f_i = \begin{cases} 1/s_i & \text{truncation,}\quad s_i > \alpha s_1 \\ 0 & \text{truncation,}\quad s_i \leq \alpha s_1 \\ s_i/(s_i^2 + (\alpha s_1)^2) & \text{Tikhonov damping} \end{cases} \tag{3}$$

with α the regularization amount, relative to the largest singular value (Schultz 2019). A truncation amount of 10^{-3} keeps the inverse's condition number under 1,000. Schultz found input estimation errors growing large where the condition number exceeded about 1,000, and that too little or too much regularization both add error (Schultz 2019). In the code, Equation 3 is `filter` and `inverse` in `crates/hertzforge-mimo-laws/src/support.rs`. Every rule first drops singular values at or below $\max(n_c, n_d)\,\epsilon\, s_1$, with n_c and n_d the numbers of control gauges and shakers and ϵ the machine precision, so an exactly singular line inverts nothing it cannot.

2. The Six Laws

Table 1 lists the six laws. Four of them (the default, the shape-constrained, the buzz-test and the minimum-drive laws) run the same closed loop and differ in what they feed it. That loop is described first. Unless a path is given, files are in `crates/hertzforge-mimo-laws/src/` of Hertzforge at main commit b587d5a1.

Table 1. The six control laws Hertzforge ships for multi-shaker random vibration.

Id	Law	Source	What it changes	File
`pinv_closed`	Pseudo-inverse, closed loop (the default)	Smallwood (Smallwood 1999); regularization after Schultz (Schultz 2019)	The shared closed loop on the full system	`pinv.rs`
`pinv_open`	Pseudo-inverse, open loop	The same	No feedback: Equation 2 every loop	`open_loop.rs`
`shape_svd`	Shape-constrained	Schultz and Avitabile (Schultz and Avitabile 2019, 2020)	Drives only the system's strongest directions on each line	`shape.rs`
`buzz_match`	Buzz-test matched	Daborn (Daborn 2014), as stated by Schultz (Schultz 2019)	Takes the target's coherence and phase from the identification	`buzz.rs`
`min_drive`	Minimum drive	Coletti, Davis and Schultz (Coletti et al. 2026); Smallwood (Smallwood 2007; Smallwood 2013)	Chooses the cross terms that need the least drive	`min_drive.rs`, `pareto.rs`
`band_average`	Band average	Brandon Zwink	One flat drive per fractional-octave band, matched on band averages	`band_average.rs`, `crates/hertzforge-law-band-average-math`

2.1 The Shared Closed Loop

What it does. A closed loop measures what the last drive produced and corrects the next one. Correcting the drive matrix directly can make it ask for negative power, which cannot be played. Smallwood's method avoids that by never correcting the drive matrix itself (Smallwood 1999). It splits the target into a triangular factor, keeps a running copy of that factor, and corrects the copy from the error each loop; the drive is always formed from the copy in a way that cannot ask for negative power.

The problem. On each line, find a drive G_{uu} that can be played and whose response matches the target:

$$G_{uu} = A\,Z Z^H A^H, \qquad H A Z Z^H A^H H^H = G_{ref} \tag{4}$$

where Z is a lower-triangular matrix with real, non-negative diagonal (g/√Hz) and A maps the factor's sources to the drives (V/g). At the start $A = H^+$ and $Z = R$, the factor of the target, $G_{ref} = RR^H$, so the first drive is Equation 2. A matrix of the form BB^H never asks for negative power (Appendix A.3), so every drive the loop makes can be played.

The derivation. Each loop compares the measured response with the target, $E = G_{ref} - G_{yy}$ (g²/Hz), and looks for the change Δ in the factor that would remove that error. When the measured frequency response is exact and the system has as many independent directions as gauges, $G_{yy} = ZZ^H$, and $(Z + \Delta)(Z + \Delta)^H = G_{ref}$ to first order in Δ is

$$\Delta Z^H + Z\Delta^H = E \tag{5}$$

With Δ lower triangular and its diagonal real, Equation 5 is solved one column at a time, the diagonal first (Smallwood 1999) (Appendix A.4):

$$\Delta_{jj} = \frac{E_{jj} - 2\sum_{m<j} \mathrm{Re}(\Delta_{jm}\overline{Z_{jm}})}{2Z_{jj}}, \qquad \Delta_{kj} = \frac{E_{kj} - \sum_{m<j}(\Delta_{km}\overline{Z_{jm}} + Z_{km}\overline{\Delta_{jm}}) - Z_{kj}\Delta_{jj}}{Z_{jj}}, k > j \tag{6}$$

This is the **first form** of the correction, which changes the size and the phase of each term. Columns are solved in channel order, so autospectra come before the cross terms they feed, and the first control gauge has the highest priority. The update is $Z \leftarrow Z + e\,\Delta$, with the convergence factor e between 0 and 1.

A cross term of the factor is not allowed to grow when its gauge is already above its target. In that case the **second form** replaces the first: it changes only the term's size, toward zero,

$$\Delta_{kj} = \begin{cases} Z_{kj}\,E_{kk}/(2\,G_{ref,kk}) & \text{when } -E_{kk} < G_{ref,kk}/2 \\ -Z_{kj}/2 & \text{when } -E_{kk} \geq G_{ref,kk}/2 \end{cases} \tag{7}$$

so a gauge more than half again over its target has the term moved halfway to zero before the convergence factor applies. This is the rule the build uses, not a derived step: if every term of the row were scaled by $1 + \delta$, the gauge's response would scale by about $1 + 2\delta$, and $\delta = E_{kk}/(2G_{ref,kk})$ would remove the relative excess to first order; the build applies that δ to each cross term the second form governs, while the row's diagonal takes the first form of its own column (Appendix A.4). Smallwood gives the second form as his Equation 13, with twice the row index times the column gauge's target in the denominator and the column gauge's error in the numerator. His appendix follows that form in the first column, and in the other columns uses the column index and the row gauge's error and target; it moves the term by half of itself when the gauge is half again off its target, and uses the second form only for a cross term that is not zero (Smallwood 1999). Which of the two forms is used in the first column changes the result on the case of Figure 3 (Section 6). The build divides by $2\,G_{ref,kk}$ in every column and applies the second form to a cross term at zero as well (Table 2).

The factorization allows a zero on its diagonal, which Smallwood calls a Cholesky factorization with zeros, citing Dodds and Robson (Smallwood 1999; Dodds and Robson 1975): a gauge whose target is fully coherent with gauges before it adds no source of its own. Smallwood sets the whole row and column of such a pivot to zero. The build counts a pivot at or below 10^{-10} of its gauge's target autospectrum as zero and sets only that column to zero for the whole run, so the gauge keeps the part of its target it shares with the gauges before it.

The coordinates. When the system carries fewer independent directions than there are control gauges (more gauges than shakers, or directions dropped by regularization or by the shape constraint), only part of the error can be corrected. The shipped loop then runs in the coordinates of what the system can produce. With q the number of directions kept, $T = U_q^H$ reduces a gauge-space matrix to q by q, and $A = V_q\ \mathrm{diag}(f_q)$ maps the reduced sources to the drives. The loop factors and corrects $TG_{ref}T^H$ against $TG_{yy}T^H$. That reduced loop is square, so on a system whose measured frequency response is exact it settles at the least-squares response $PG_{ref}P^H$, the response the open-loop pseudo-inverse gives (Appendix A.5). q counts the singular values the inverse kept that are also above the rank tolerance (by default 10^{-6}) times the largest. In reduced coordinates,

"gauge k" in the rules below reads "source k": E_{kk}, $G_{ref,kk}$ and the hold act on the diagonal of the reduced matrices.

The algorithm as built. `FactorLoop` in `factor.rs`:

1. `FactorLoop::seed`: form the inverse (`support.rs`, `inverse`); per line, choose the coordinates (`LineMap::new`), reduce and factor the target (`factor_with_zeros`), set $Z = R$, and return $G_{uu} = (AZ)(AZ)^H$ (`FactorLoop::drive`, a Gram matrix by `gram_packed`).
2. `FactorLoop::iterate`, per line: if the measured frequency response changed since the last call, rebuild the coordinates and carry the factor over by factoring the law's own last drive in the new coordinates, $A^+ G_{uu} A^{+H}$, so the correction made so far is kept (`FactorLoop::refresh`).
3. Reduce the measured response and form E. A line with no target is skipped.
4. Solve Equation 6 column by column, replacing a cross term by Equation 7 where its gauge is over target and the first form would grow it. The divisor is $\max(Z_{jj}, Z_{min})$.
5. **Diagonal limit.** No diagonal term of the factor may move by more than the per-loop limits of power, by default +3.5 dB and −6 dB (a factor element moves by half as many decibels). A column whose diagonal would move more has its first-form terms scaled down by the diagonal's own ratio; second-form terms are applied as they are.
6. **Hold on a source that does not answer.** When a source's measured response, in the loop's coordinates, is below 0.03 (-15 dB) of what its row of the factor commands, $(TG_{yy}T^H)_{kk} < 0.03\,(ZZ^H)_{kk}$, no element of that row may grow in that loop; each may still shrink.
7. Update $Z \leftarrow Z + e\Delta$.
8. **Bound on what a source commands.** No row may command more than ten times (+10 dB) its source's target, $(ZZ^H)_{kk} \le 10\,(TG_{ref}T^H)_{kk}$, or more than it commanded the loop before where that was already higher; a row over the bound is scaled down to it.
9. **Floor.** No diagonal term falls below $Z_{min} = 10^{-4}\sqrt{\max_k (TG_{ref}T^H)_{kk}}$; when one would, it is set to the floor, and the rest of its column is set to its values before this loop's update, scaled by the ratio of the new diagonal to the old (clamped to 0 to 1); this loop's correction to them is dropped.
10. **Drive limit.** The step is halved, line by line, until no shaker's drive autospectrum moves by more than the per-loop limits from the law's last drive (a drive under 10^{-6} of the line's largest is measured against that floor). After 30 halvings the line's factor is held where it was.
11. Return $G_{uu} = (AZ)(AZ)^H$, with the diagnostics `condition`, `kept`, `correction_db` (the largest change of any drive autospectrum, dB) and `clamped` (diagonals held by step 5).

Settings. The convergence factor, the two per-loop limits and the rank tolerance are settings of the default law (Table 3); the other three laws that use this loop fix the limits at +3.5 and −6 dB and the rank tolerance at 10^{-6} (`ShapeConstrainedLaw::new`, `BuzzMatchedLaw::seed`, `MinimumDriveLaw::seed`).

Edge cases.

- *As many shakers as gauges, or fewer, at full rank:* the loop runs in the gauges' own coordinates with $A = H^+$, exactly as the source states it.
- *More gauges than shakers, or a line where the inverse dropped directions:* the reduced coordinates above; the error outside what the system can produce is not corrected.
- *A rank-deficient or nearly singular line:* the directions the inverse dropped are not sources; if no direction is kept, $q = 0$ and the line's drive is zero.
- *A tiny singular value that survives truncation:* the hold (step 6) and the bound (step 8) stop the loop from raising the drive without limit where the structure does not answer.
- *A line with no target:* skipped; its drive is zero.

- *The first loop:* the seed is the pseudo-inverse drive; the first correction uses the first measured response the controller hands on, after the settling loop and the soft start's climb.

Source and differences. D. O. Smallwood, "Multiple Shaker Random Vibration Control: An Update" (Smallwood 1999), read in full, with its Appendix A, the correction as a MATLAB function, read line by line from the scan. Table 2 lists every place the build differs from that appendix, from `factor.rs` at b587d5a1. His paper also describes rescaling the diagonal corrections when they become small against Z_{jj}, because division by a large Z_{jj} had left it stuck high, and warns that a term of the factor at zero can stay stuck there. The code's documentation gives the measurements behind the build's changes (`factor.rs`, module documentation).

Table 2. Where the shared loop as built at b587d5a1 differs from the correction in Smallwood's Appendix A and his factorization with zeros (Equation 8), read from the scan of SAND98-2044C.

Item	Smallwood's appendix	The build
Floor	10^{-4} times the largest element of the target, a power	10^{-4} times the square root of the largest target autospectrum, in the factor's units
Convergence factor	One value, or one per element	One value
Rank	Rows and columns of the factor past the rank of the measured frequency response are zeroed	The loop runs in the reduced coordinates of the directions kept
First diagonal	Divided by the first diagonal term, with no floor	Divided by the term, never below the floor
Divisors	The larger of the first and the column's diagonal term	The column's diagonal term, never below the floor
Second-form step	First column, as Equation 13: twice the row index times the first gauge's target, with the first gauge's error; other columns: twice the column index times the row gauge's target, with its error	Twice the row gauge's target, with its error, every column
Second form on a zero term	Not used: only for a cross term that is not zero	Used, so a term at zero stays there
Zero cross term	Set to the floor after the update	Left at zero
Order of the forms	The first form is solved whole; the second replaces terms after the update, and later columns use first-form values	The second form replaces terms inside the solve, and later columns use its values
Diagonal limit	Each diagonal correction at most half of the larger of its own and the first diagonal term, alone	+3.5 dB and −6 dB of its own term, with the column's first-form terms scaled alike
Diagonal at zero	Set to the floor when zero or negative, from the second column on; a negative first term stops the solve	Every live diagonal under the floor is set to it
Column collapse	A column whose diagonal is under the floor has its cross terms set to a hundredth of the floor, phase kept	Its cross terms are set to their previous values times the ratio of the new diagonal to the old
Zero pivot	Its whole row and column are zero at the factorization (Equation 8)	Its column only, below 10^{-10} of the gauge's target
Dead column	The next correction gives the zeroed pivot and its column values again	Held at zero for the whole run
Large correction	A warning above 10^{6}	No check
Not in the source		The hold, the bound, the drive limit with halving, the carry-over on a new measured frequency response, weighting and regularization of the inverse

The hold and the bound are not in the source. They were added for a case the code's documentation records: a lightly damped plate with 48 control gauges and 12 shakers, identified through measurement noise, whose drive rose loop after loop in directions the measured frequency response held and the structure did not answer.

What it gives up by design. The drive can change only by the per-loop limits, so a large correction takes several loops. Where the system answers less than −15 dB of what a source commands, the loop stops raising that source, and no source commands more than +10 dB over its target, so a direction in which the structure is more than 10 dB less sensitive than measured is not fully corrected.

Figure 3 shows the loop at work on a made-up structure with two gauges and two shakers, whose measured frequency response has every mode 3 percent too high, against the open-loop drive of Equation 2. It was made by running the shipped `pinv_closed` and `pinv_open`, at their defaults, with an exact measurement of the response and no noise (`code/shipped_laws`).

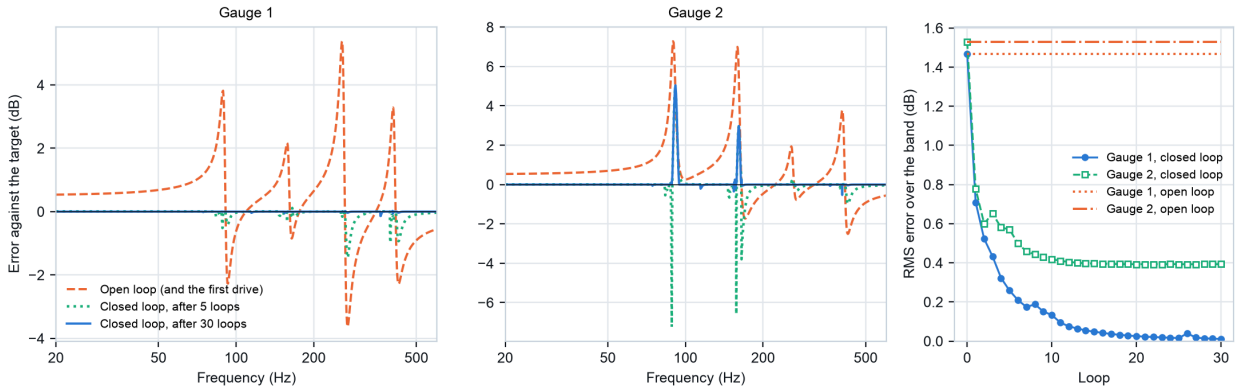


Figure 3: The shipped default law correcting a measured frequency response that is 3 percent off, on a made-up two-gauge, two-shaker structure, against the open loop. Left and middle, the error at each gauge line by line; right, the RMS error over the band, loop by loop. An illustration, not a result about how the law scores.

At gauge 1 the loop removes the error. At gauge 2 it removes most of it and stops short on a few lines near two resonances, where the gauge stays a few decibels high. On those lines the measured frequency response is off near a mode, so the drive meant for gauge 1 reaches gauge 2 as well, and gauge 2 reads above its target. The one term of the factor that could cancel that leak is the cross term Z_{21}, and cancelling it means making the cross term larger. The second form forbids exactly that: a cross term whose gauge is above its target may only shrink. The loop then cuts gauge 2's own diagonal term by the per-loop limit, loop after loop, until it reaches the floor, and gauge 2 is left reading the leak alone. Appendix B.8 follows this on the line at 91 Hz, where gauge 2 ends 4.87 dB high. Smallwood's second form, as published, can hold a gauge high in the same way: a port of his Appendix A run on the same case (`code/smallwood_port`) holds gauge 2 at the build's levels from 90.5 to 93 Hz when the target has cross terms (coherence 0.64, phase 30 degrees), and at 91.5 and 92 Hz with the independent target of Figure 3 (+5.04 and +4.61 dB at loop 30). With the independent target it releases the line at 91 Hz (+0.40 dB at loop 30, against the build's +4.87 dB), and any one of four of the build's departures from his appendix holds it again: the second form on a cross term at zero, the second-form step, the order of the two forms, or the limit on the diagonal (Table 2, Section 6). Near 160 Hz the hold is the build's own departure: the cross term starts at zero and the build applies the second form to a term at zero, so it never moves. Appendix B.3 follows a line where the term is held at zero for three loops and released when the gauge dips below its target.

2.2 Pseudo-Inverse, Closed Loop (`pinv_closed`, the Default)

What it does. It starts from the regularized pseudo-inverse drive and corrects it every loop by the shared closed loop of Section 2.1, on the full system. It is the law a run uses unless another is chosen (`DEFAULT_MIMO_LAW_ID` in `crates/hertzforge-mimo-laws/src/lib.rs`).

The problem and its derivation. Equation 4 with A the regularized pseudo-inverse of Equation 3. With error weighting on, the least-squares inverse is $(WH)^{+}W$ with $W = \mathrm{diag}(1/\sqrt{G_{ref,kk}})$, so each gauge's error counts relative to its own target; a gauge with no target is weighted as one with 10^{-12} of the line's largest (`support.rs`, `weights`). On a system with as many shakers as gauges, weighting changes nothing when no singular value is dropped or damped, since then $(WH)^{-1}W = H^{-1}$; when some are, it changes which directions are dropped.

The algorithm as built. `PseudoInverseLaw` in `pinv.rs` builds the inverse rule from its settings (`PseudoInverseConfig::rule`) and hands every call to `FactorLoop` (Section 2.1).

Table 3. The settings of `pinv_closed` (`support.rs`, `regularization_params`, `convergence_param`, `cap_params`, `rank_tolerance_param`, `weighting_param`).

Parameter	Default	Range	What it changes
`regularization`	truncated	truncated, tikhonov, none	How weak directions are treated (Equation 3)
`regularization_amount`	10^{-3}	0 to 1	The amount of Equation 3, relative to the largest singular value
`convergence`	0.5	0.1 to 1	The convergence factor: the share of each correction applied
`cap_up_db`, `cap_down_db`	3.5 dB, 6.0 dB	0.1 to 3.5 dB, 0.1 to 6.0 dB	The per-loop limits of steps 5 and 10
`rank_tolerance`	10^{-6}	0 to 0.1	Singular values counted as sources; above the truncation amount it drops sources truncation kept, at or below it nothing
`weighting`	none	none, reference	Each gauge's error equal, or relative to its target

Edge cases. As Section 2.1. With `regularization` set to none, every singular value above the numerical floor is inverted, and the rank tolerance alone decides which directions are sources.

What it gives up by design. It asks for the full target, cross terms included, wherever the system can produce it; it spends drive in weak directions up to the regularization's limit to do so. Where the structure changes with level, the measured frequency response, taken about 20 dB under the target and kept by default, is no longer the system at full level; the loop treats the difference as an error in the measured frequency response and corrects it within its per-loop limits, the hold and the bound.

Source and differences. D. O. Smallwood, "Multiple Shaker Random Vibration Control: An Update" (Smallwood 1999), read in full: the closed-loop correction, its appendix and its safeguards. The regularization choices are from Schultz (Schultz 2019), read in full. The differences are those of Section 2.1.

2.3 Pseudo-Inverse, Open Loop (`pinv_open`)

What it does. It plays Equation 2 every loop, computed from the measured frequency response and the target alone. The measured response is not used. With the frequency response measured once and kept, the drive is the same every loop; with it measured again during the run, the drive follows the new measurement.

The problem and its derivation. Equation 2 with the inverse of Equation 3 and the weighting of Section 2.2. With more gauges than shakers it minimizes the distance between target and response; with fewer it is the least-power drive that meets the target (Appendix A.2).

The algorithm as built. `OpenLoopLaw::step` in `open_loop.rs`: the cached inverse (`InverseCache::get`, formed again only when the measured frequency response changes), the congruence $H^{+}G_{ref}H^{+H}$ on the in-band lines, zero outside the band.

Table 4. The settings of `pinv_open`.

Parameter	Default	Range
`regularization`	truncated	truncated, tikhonov, none
`regularization_amount`	10^{-3}	0 to 1
`weighting`	none	none, reference

Edge cases. A rank-deficient line inverts only the directions kept; a line with no target gets no drive. There is no first-loop special case: the seed and every loop are the same computation.

What it gives up by design. Any error in the measured frequency response, including a structure that has moved with level, passes straight to the test (Figure 3, dashed). In exchange it has no memory and cannot wind up, and on a system whose measured frequency response is exact it reaches what the system can reach and no more. It is the baseline against which the closed-loop laws can be read, and the simplest law to reason about. It is also the worked example of a complete law in the law kit (Section 4).

Source and differences. As `pinv_closed` (Smallwood 1999; Schultz 2019). No differences.

2.4 Shape-Constrained (`shape_svd`)

What it does. On each line it drives only the m patterns the system responds to most, the leading columns of V, and gives up the rest. The weak directions are where a pseudo-inverse spends the most drive for the least response, so constraining the drive to the strong ones trades match between modes for drive. In the authors' simulation of a beam with four shakers, two directions matched the target response closely, three needed much more force than two, and the method was nearly as accurate as the standard inverse with much lower forces (Schultz and Avitabile 2019).

The problem. With $C = V_m$, the first m columns of V on a line, the constrained system is $H_c = HC$, the constrained drive is $G_c = H_c^{+}G_{ref}H_c^{+H}$, and the shaker drive is $G_{uu} = CG_cC^H$ (Schultz and Avitabile 2019).

The derivation. Because $HC = U_mS_m$, the drive's inverse is $CH_c^{+} = V_mS_m^{-1}U_m^H$, the pseudo-inverse truncated to exactly m directions (Appendix A.6), which the authors point out themselves (Schultz and Avitabile 2019). The law is therefore the shared loop of Section 2.1 with an inverse that keeps m directions, so no more than m independent sources are ever driven, in the seed or in any loop.

The algorithm as built. `ShapeConstrainedLaw` in `shape.rs` hands `FactorLoop` the rule `InverseRule::Shape`; `filter` in `support.rs` keeps, by default, the singular values at or above 0.2 times the largest, at most `max_shapes`, and at most the numerical rank (Figure 4). With selection by count it keeps `shapes` directions, fewer where the rank

is lower. The loop then runs in the reduced coordinates of the m kept directions whenever m is less than the number of gauges.

Table 5. The settings of `shape_svd`. The per-loop limits (+3.5 dB, −6 dB) and the rank tolerance (10^{-6}) are fixed.

Parameter	Default	Range	What it changes
`selection`	ratio	ratio, count	How the number of directions is set on each line
`ratio`	0.2	0.1 to 0.5	Keep singular values at or above this times the largest
`max_shapes`	1,024	1 to 1,024	Most directions kept, with selection by ratio
`shapes`	2	1 to 1,024	Directions kept, with selection by count
`convergence`	0.5	0.1 to 1	The shared loop's convergence factor

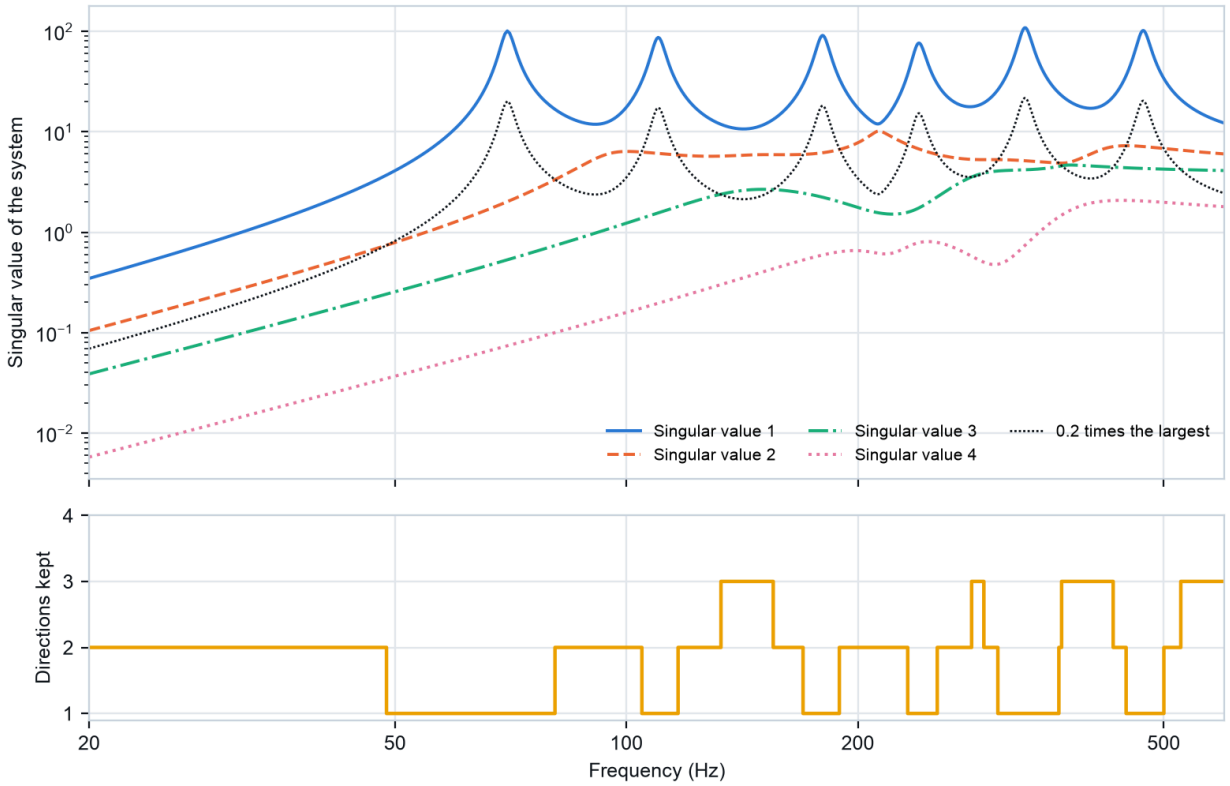


Figure 4: How many directions the shape-constrained law keeps on each line of a made-up structure with six gauges and four shakers, at its default ratio of 0.2. Computed from the law's selection rule and checked against the shipped law. An illustration, not a result.

Edge cases. At least one direction is always kept on a line with any response, since the largest singular value passes the ratio. On a rank-deficient line, m is at most the rank. Where two singular values cross from one line to the next, the kept direction can swap between neighboring lines. The condition number reported is that of the whole system, not of the constrained one.

What it gives up by design. Between modes, where several singular values are of similar size, a small m leaves out response the target needs; in the authors' example a single direction matched well near the peaks and poorly between them (Schultz and Avitabile 2019). It does not try to meet cross terms outside the kept directions.

Source and differences. R. Schultz and P. Avitabile's conference paper, "Shape-Constrained Input Estimation for Multi-Shaker Vibration Testing" (Schultz and Avitabile 2019), read in full; the journal version (Schultz and Avitabile 2020) was not obtained, and nothing is said here of its content. The ratio rule for m is Hertzforge's. The closed loop inside the constraint is Hertzforge's addition; the source states the open-loop drive.

2.5 Buzz-Test Matched (`buzz_match`)

What it does. It keeps the target's autospectra and replaces the target's cross terms with the coherence and phase the article shows on its own, under the uncorrelated drive of the identification. Those cross terms are ones the structure produces of itself. The modified target is formed once, at the start, and the closed loop of Section 2.1 then holds it.

The problem. With B the identification's response matrix on a line, the coherency of a pair is $C_{jk} = B_{jk}/\sqrt{B_{jj}B_{kk}}$ and the modified target is

$$G_{jk} = \sqrt{G_{ref,jj}\,G_{ref,kk}}\,C_{jk} \tag{8}$$

that is, $G = DCD$ with $D = \mathrm{diag}(\sqrt{G_{ref,jj}})$. A pair whose identification coherence $|C_{jk}|^2$ is under a floor is left independent, $G_{jk} = 0$ (Figure 5).

The derivation. C is positive semi-definite because B is, so DCD is too (Appendix A.7). Leaving pairs out can break that and make a line ask for negative power, so each line is then repaired by clamping its negative eigenvalues, after the repair of Hunter, Cross and Nelson (Hunter et al. 2018), and then restoring the autospectra by a scaling of rows and columns.

The algorithm as built. `BuzzMatchedLaw::seed` in `buzz.rs`: `buzz_reference` forms Equation 8 on the in-band lines from the identification response the controller hands the law (it declares the input `SysidResponse`), leaves a pair independent when either autospectrum of the identification is zero or the coherence is under the floor, then repairs every negative eigenvalue however large (`linalg::psd_repair` in `crates/hertzforge-sdm`). It then builds `FactorLoop` on the modified target with truncation at 10^{-3}, no weighting, and the fixed per-loop limits. `iterate` hands every later call to the loop.

Table 6. The settings of `buzz_match`. The loop's regularization is truncation at 10^{-3} and its per-loop limits are the default law's.

Parameter	Default	Range	What it changes
`coherence_floor`	0.1	0 to 1	Pairs whose identification coherence is under this are left independent
`convergence`	0.5	0.1 to 1	The shared loop's convergence factor

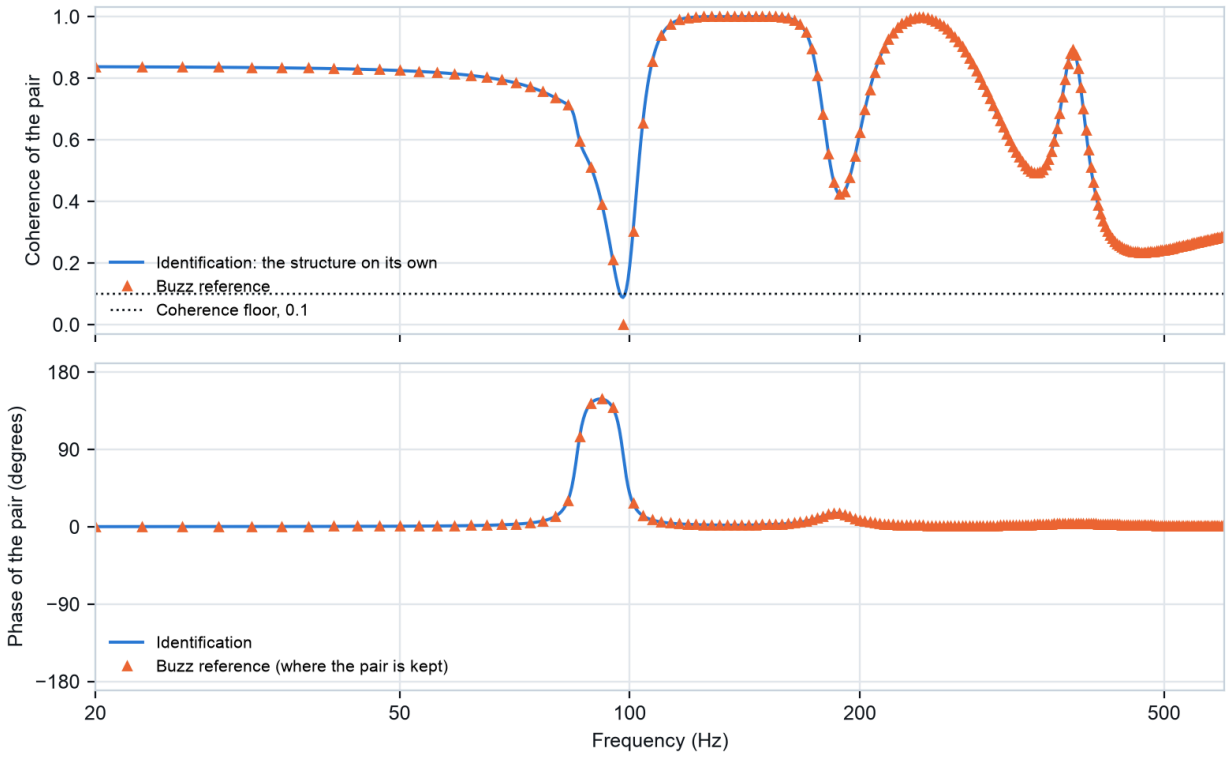


Figure 5: The buzz-test target of one pair of gauges on a made-up structure: the coherence and phase it takes from the identification, and the floor under which the pair is left independent. Checked against the shipped law. An illustration, not a result.

Edge cases. The target is formed from the identification as measured, at about 20 dB under the target, and is not formed again if the frequency response is measured again during the run. A pair left independent starts its factor's cross term at zero, so the behavior of a cross term at zero in Section 2.1 and Section 6 applies to this law by design. The law declares that it needs the identification response, and the controller supplies it; called without it, `seed` returns an error.

What it gives up by design. It does not try to meet the specified cross terms. It asks only for the coherence and phase the structure shows on its own under uncorrelated drive. The drive this needs is not in general uncorrelated (Appendix B.5). Where the structure changes with level, the coherence and phase it holds are those of the structure at the identification's level.

Source and differences. The method is known here as Schultz states it (Schultz 2019) (read in full): the target's cross terms are replaced by the coherence and phase the structure shows under uncorrelated white noise on the shakers, a method he describes as used by P. M. Daborn and for which he cites Daborn's thesis (Daborn 2014), not read. Coletti and co-authors describe the same method as the independent drives method, also called the buzz method, which they attribute to Smallwood (their reference 16) (Coletti et al. 2026). The repair is after Hunter, Cross and Nelson (Hunter et al. 2018), read in full, who set a negative eigenvalue to a small positive number and rebuild the matrix; the build clamps every negative eigenvalue and then restores the autospectra, which is Hertzforge's. Holding the modified target fixed and running the full closed loop on it is Hertzforge's; Schultz uses the modified target in an open-loop estimate of the inputs.

2.6 Minimum Drive (`min_drive`)

What it does. When the specification fixes only autospectra, the cross terms are free, and some choices need far less drive than others. This law chooses them. On each line it finds the drive that best trades two things against each other, the error in the autospectra and the total drive power, and makes the response the system would produce under that drive its target. The closed loop of Section 2.1 then holds that target.

The problem. On each line, with y_k the specified autospectra (g²/Hz) and w_k their weights, the drive X (V²/Hz) minimizes

$$\sum_k w_k \left| y_k - (HXH^H)_{kk} \right| + \lambda \operatorname{tr} X \quad \text{over positive semi-definite} \quad X \tag{9}$$

where $\operatorname{tr} X$ is the total drive power and λ the price of a unit of drive (Coletti et al. 2026). The target is then HXH^H, the full matrix, cross terms included. By default the weights are $w_k = 1/y_k$, so each gauge is judged by the fraction of its spectrum it misses, and the weighted error is dimensionless. As λ falls toward zero the solution becomes Smallwood's minimum input trace: of all drives that meet the autospectra exactly, the one of least total power (Smallwood 2007; Smallwood 2013). Figure 6 (left) shows the principle on one line: with the autospectra held, the drive power depends on the cross term's coherence and phase, and the least lies at full coherence and one phase (Appendix A.8).

The derivation. Equation 9 is convex. Writing each weighted absolute value as $w_k|r_k| = \max_{|\nu_k| \le w_k} \nu_k r_k$ and exchanging the minimum and the maximum gives its dual (Appendix A.8): maximize $\sum_k \nu_k y_k$ over real ν, subject to $S(\nu) = \lambda I - H^H \operatorname{diag}(\nu) H$ positive semi-definite and $|\nu_k| \le w_k$. The dual has one real unknown per gauge where the primal has the square of the number of shakers. With a logarithmic barrier of weight μ, the minimizing drive is $X = \mu S^{-1}$, and the barrier's residuals vanish as μ does.

The trade-off is given relative to the line, $\lambda = \lambda_{rel}\, r_0/d_0$, with $r_0 = \sum_k w_k y_k$ the weighted error of no drive at all and $d_0 = \sum_k y_k / \|h_k\|^2$ a drive power scale (h_k is gauge k's row of H; $y_k/\|h_k\|^2$ is the least drive power that puts y_k on gauge k alone), so the same λ_{rel} means the same on every system.

By default λ_{rel} is chosen automatically. The grid $\lambda_{rel} = 10^{-6}, 10^{-5}, \ldots, 10^{6}$ is solved on a sample of the band's lines, and for each grid point g the drive power $D_g = \sum \operatorname{tr} X$ and weighted error R_g are summed over the sample. A grid point is left out when D_g or R_g is above nine tenths of its largest value over the grid, when its target leaves the specification's alarm lines on any sampled gauge and line, or when its target keeps, on any one gauge, less than $10^{-\text{ shortfall}/10}$ of the specified autospectrum on average over the sampled lines, each line counted at most at its specification. Of the points left, the one with the smallest

$$(D_g + \varepsilon)(R_g + \varepsilon), \qquad \varepsilon = 10^{-12}(\max D + \max R) \tag{10}$$

is taken (Coletti et al. 2026, sec. 3.1, with the product in place of a sum of logarithms). If no point is left, the smallest trade-off is taken. The **largest shortfall** (`shortfall_db`, 3 dB by default) is a floor for every gauge, so no gauge is given up for the others.

The algorithm as built. `MinimumDriveLaw::seed` in `min_drive.rs` calls `pareto::target`:

1. `Lines::new` (`pareto.rs`): per in-band line, the measured frequency response, the specified autospectra (negative values read as zero), the weights (`residual_weights`: $1/y_k$, a gauge with no target weighted as 10^{-12} of the line's largest), the scales r_0, d_0 (`scales`) and the alarm lines.
2. `choose_tradeoff` (automatic only): the sample is every $\lceil N/64 \rceil$-th of the N in-band lines, from the first, at most 64 lines; each sampled line is solved at every grid point, each solve started from the last one's dual point; then the rule of Equation 10.

3. `solve_line`, every in-band line at the chosen λ_{rel}: the dual by damped Newton steps on the barrier (step $1/(1+\text{decrement})$, halved until the point stays inside), μ divided by 8 between stages, stopping when the barrier's bound on the duality gap, $\mu(n_d + 2n_c)$, is under 10^{-7} of the objective; then $X = \mu S^{-1}$.
4. `clean_rank`: the drive's eigenvalues under 10^{-3} of its largest are set to zero, so the target has the rank of its solution and the loop's factor has no small, arbitrary sources.
5. The target HXH^H on every in-band line, with the diagnostics `tradeoff`, `drive_power` (per line) and `target_ratio` (the target's autospectrum over the specified one, per gauge and line: what the trade-off gave up).
6. `FactorLoop` on that target, truncation at 10^{-3}, no weighting, fixed per-loop limits; every later call goes to the loop.

Table 7. The settings of `min_drive`. The loop's regularization is truncation at 10^{-3} and its per-loop limits are the default law's.

Parameter	Default	Range	What it changes
`tradeoff`	auto	auto, fixed	Chosen by Equation 10, or set
`tradeoff_value`	1	10^{-6} to 10^{6}	The relative trade-off, when fixed
`shortfall_db`	3 dB	0 to 20 dB	The largest shortfall, when automatic
`residual`	relative	relative, absolute	Each gauge's error relative to its specification, or in its own units
`convergence`	0.5	0.1 to 1	The shared loop's convergence factor

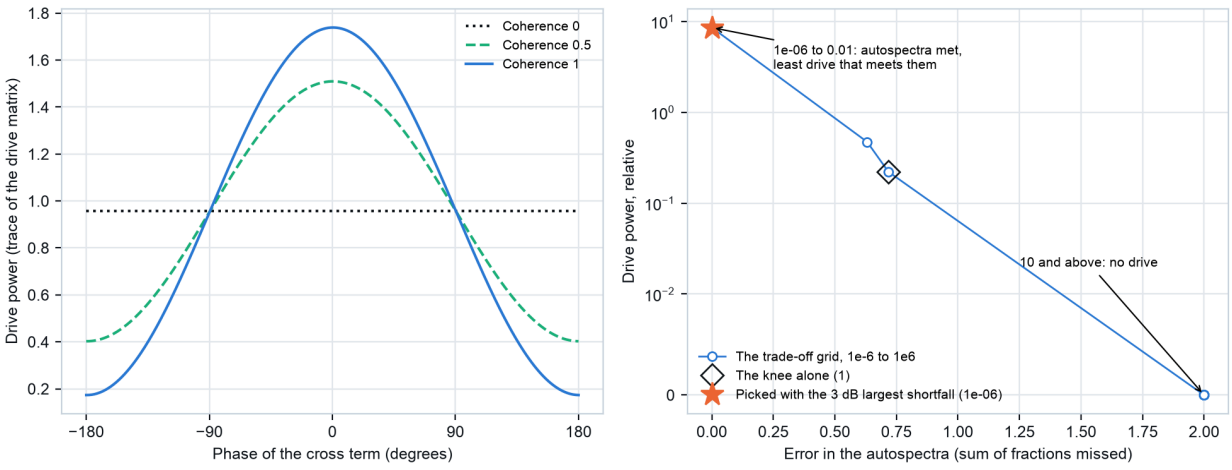


Figure 6: The minimum-drive target on one line (200 Hz) of a made-up structure with two gauges and two shakers. Left: the drive power a target needs, its autospectra held, as its cross term's phase and coherence change. Right: drive power against the error in the autospectra along the trade-off grid, with no alarm lines, the point the knee of the curve alone would pick, and the point the automatic rule picks with its default largest shortfall of 3 dB. Checked against the shipped law. An illustration, not a result.

Edge cases. A line with no target, or whose measured frequency response is zero at every gauge, gets no drive (r_0 or d_0 is zero and the line is not solved). With fewer gauges than shakers the exact-control end is the minimum-norm drive; with more, the autospectra are met where the system can and the error term takes the rest. The target is formed once, from the identification's measured frequency response; it is not formed again if the frequency response is measured again during the run, though the loop's coordinates follow the new measurement. The rank cut of step 4 applies only to the target; the automatic choice reads the uncut solutions.

What it gives up by design. With the trade-off above its exact-control end, the target's autospectra sit below the specification where the system makes them costly, by the ratio `target_ratio` reports. The largest shortfall is an average over the sampled lines, so a gauge may sit up to 3 dB under its specification on average by default, and further on single lines; the alarm lines, where a test gives them, bound single lines. Without alarm lines, the knee of the curve leaves out the grid points whose drive power is near the largest, and on a line where every exact-control point needs about the same drive that leaves out all of them: on the line of Figure 6 the knee alone takes a point that keeps 28 percent of gauge 1′s specified spectrum. The largest shortfall leaves that point out, and the rule takes the exact-control end. The code's documentation records the case the largest shortfall was added for: on the 48-gauge plate of Section 2.1, with no alarm lines, the knee alone chose a target keeping 15 percent of the specification (`pareto.rs`, `choose_tradeoff`).

Source and differences. K. Coletti, R. B. Davis and R. Schultz, "Pareto-optimal target definition for multi-axis random vibration testing" (Coletti et al. 2026), read in full in its preprint form (SAND2026-18747J): the problem (their Equation 2, unweighted), its special cases and the automatic choice (their Section 3.1: the least sum of the logarithms of residual and drive power over a logarithmic grid such as $\{0, 10^{-6}, \ldots, 10^{6}\}$, after leaving out the highest 10 percent of each). D. O. Smallwood's extreme inputs and outputs (Smallwood 2007), read in its report form; his minimum input trace paper (Smallwood 2013), not read. The minimum drives method of Musella and co-authors (Musella et al. 2018), known through Coletti and co-authors, not read. The residual weights, the dual form, the damped Newton solve, the relative scaling of the trade-off, the product in Equation 10, the alarm-line and largest-shortfall rules and the rank cut are Hertzforge's.

2.7 Band Average (`band_average`)

What it does. It matches averages over fractional-octave bands, one-sixth octave by default, instead of single lines. In each band it solves for one drive matrix, the same on every line of the band, whose band-averaged response matches the band-averaged target. The matrix is fitted whole, cross terms and all, so the drives come out correlated, uncorrelated or partly correlated as the fit needs. Averaging the frequency response over a band before inverting it keeps the noise in the identification from being amplified line by line. A threshold per band, chosen once at the first solve, sets how far each band's inverse is regularized. After that a gain correction, band by band and gauge by gauge, moves each band's target by the ratio of predicted to measured response. The law is Brandon Zwink's.

The problem. For band b (edges f_l, f_u, width $f_u - f_l$ in Hz) and control gauge p, the response to a drive matrix P_b (V²/Hz) held flat across the band, averaged over the band, is

$$\bar{y}_{b,p} = \frac{1}{f_u - f_l} \sum_{k \in b} w_k \, h_p(k) \, P_b \, h_p(k)^H \tag{11}$$

where $h_p(k)$ is gauge p's row of H on line k and w_k (Hz) is how much of line k's bin lies inside the band. Equation 11 is linear in the entries of P_b; its coefficients are the **band sensitivity**. Stacking the gauges (and, when cross terms are fitted, the band-averaged cross terms) gives a set of linear equations $A_b \, \mathrm{vec}(P_b) = t_b$, with t_b the band-averaged target $t_{b,p} = \sum_k (w_k/(f_u - f_l)) \, G_{ref,pp}(k)$.

The derivation. The equations are solved through the singular value decomposition of A_b with the Tikhonov filter

$$\mathrm{vec}(P_b) = V\ \mathrm{diag}(\phi_i)\, U^H t_b, \qquad \phi_i = \frac{s_i}{s_i^2 + (s_1/\kappa_b)^2} \tag{12}$$

where κ_b is the band's threshold: the larger it is, the closer the solve is to the plain pseudo-inverse. The solution is made Hermitian and then positive semi-definite by dropping negative eigenvalues. When only autospectra are fitted and that moves a predicted response by more than 0.1 dB, the band is fitted again over drives that ask for no negative power.

The thresholds are chosen at the first solve by a sweep. Every band is solved at 81 values of κ from 1 to 10^5 (65 values from 10 to 10^5 for the knee rule), and each band's predicted mean decibel error over its gauges, $\bar{e}_b(\kappa)$, and drive energy, $\mathcal{E}_b(\kappa) = \mathrm{tr}\ P_b \times \sum_k w_k$, are recorded. By default ("front") one common price ρ on drive energy is set for every band: each band takes the κ that minimizes $n_b \bar{e}_b(\kappa) + \rho\, \mathcal{E}_b(\kappa)$, with n_b its number of gauges with a target, so each band spends drive where it buys the most reduction in error. The price is swept over 241 values from 10^{-6} to 10^6 times a reference price, and of the prices whose mean error over every gauge in every band is within 0.1 dB of the best, the one needing the least total energy is kept. The thresholds are then held for the run.

The gain correction compares, band by band and gauge by gauge, the measured response with the response the band sensitivity predicts from the drive actually played, and smooths the logarithm of their ratio,

$$\ell_{b,p} \leftarrow (1 - r)\, \ell_{b,p} + r \ln \frac{\hat{y}_{b,p}}{y_{b,p}}, \qquad |\ell_{b,p}| \le \ln 10^{0.6} \tag{13}$$

where $\hat{y}_{b,p}$ is the band average the band sensitivity predicts from the drive played and $y_{b,p}$ the measured band average, at a rate r of 0.25 per call, capped at ±6 dB. The band's target for that gauge is multiplied by $e^{\ell_{b,p}}$, and the bands are solved again when any multiplier has moved by more than 0.05 dB since the last solve. Comparing with the prediction rather than with the target means a change of drive the law made on purpose is never fought.

Figure 7 shows the first solve on one gauge and one shaker: the band averages meet the target, the lines inside a band do not, and in the one band where the drive is costliest the common price leaves the band 1.9 dB under its target.

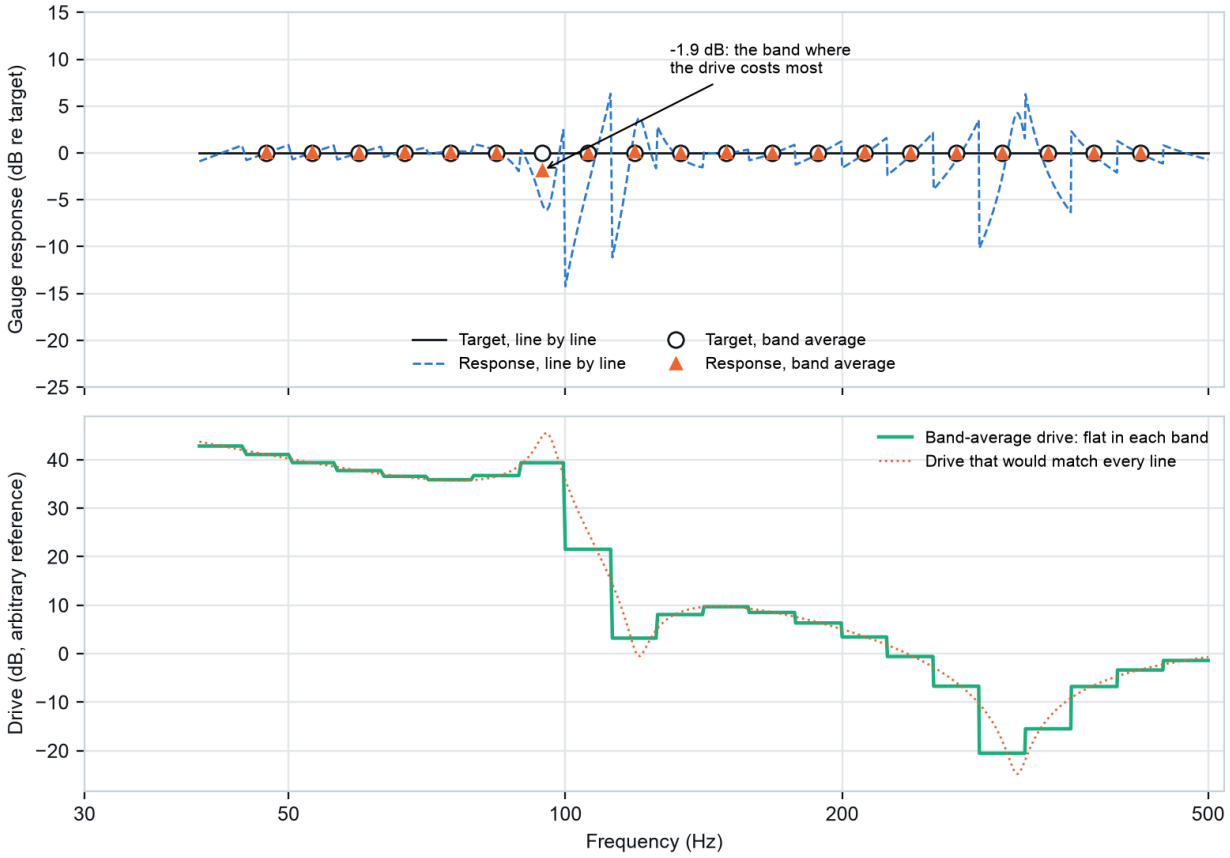


Figure 7: The shipped band-average law's first solve, at its defaults, on a made-up structure with one gauge and one shaker. Top, the response line by line and band by band against a flat target; bottom, the drive. An illustration, not a result.

The algorithm as built. `BandAverageLaw` in `band_average.rs` carries the inputs across the law boundary; the arithmetic is `BandAverage` in `crates/hertzforge-law-band-average-math/src/lib.rs`:

1. `BandAverage::initialize`, at the seed: the band edges, base ten, referred to 1 kHz (`bands.rs`, `nth_octave_edges`); each band's target averages t_b over every line whose bin overlaps the band; each band's lines for the sensitivity are those whose bin center lies in it and where some gauge has a target, each weighted by its overlap w_k. With cross terms on, every pair whose target cross spectrum is non-zero on some line becomes a row.
2. `build_sensitivity` and `band_sensitivity`: A_b of Equation 11 from the measured frequency response.
3. `solve_octave`: at the first solve, the sweep (`sweep.rs`, `resolve_auto_kappa`); if no band can be scored, a threshold of 300 for every band. Then `solve_bands`, per band `solve_band`: `fit_band` (Equation 12 with `filter`), `projected` (negative eigenvalues dropped), `realizable` (the re-fit, `fit.rs`, `psd_fit`, when only autospectra are fitted and the drop moved a response by more than 0.1 dB). `expand` writes P_b on each of the band's lines; other lines get no drive.
4. On each later call (`BandAverage::control`): `measured_predicted` (band averages of the measured autospectra and of the response the band sensitivity predicts from the played drive), `check_channels`, `update_gain` (Equation 13), and a new solve when a multiplier moved more than 0.05 dB; otherwise the last drive is returned.
5. With the frequency response update on, and the controller's estimate holding at least `estimate_frames` frames: the band sensitivity is formed again from the controller's updated frequency response and averaged into the old one, the drive is conditioned with a diagonal load so the estimate sees every drive direction (bounded

by `drive_condition_number` and `min_drive_fraction`, and capped at half of the smallest ratio of a target to its band sensitivity's trace), and the bands are solved again.

Table 8. The settings of `band_average` for the solve and the gain correction (`band_average.rs`, `parameters`). The knee rule takes the first threshold within 0.1 dB of the band's best error.

Parameter	Default	Range	What it changes
`oct_order`	6	1 to 48	Bands per octave
`kappa_rule`	front	front, knee, lcurve	How the sweep picks each threshold: the common price, each band's knee, or its L-curve corner
`drive_reduction`	0	0 to 0.99	Share of the drive energy to give up, along the front
`regularization`	auto	auto, none	The sweep, or the plain pseudo-inverse in each band
`condition_number`	0	0 to 10^{12}	A fixed threshold for every band; 0 runs the sweep
`constrained_solve`	on	on, off	The re-fit over drives that ask for no negative power
`gain_correction`	on	on, off	The gain correction of Equation 13
`gain_correction_rate`	0.25	0 to 1	Its rate
`channel_check_db`	20 dB	0 to 200 dB	How far from its prediction a gauge is left out of the correction and the update; 0 is off
`drive_rms_limit`	0 (off)	0 to 10^{9} V	Hold each drive under this RMS by moving along the front
`cross_terms`	off	off, reference, buzz	Fit the autospectra only, the target's cross terms too, or the buzz-test cross terms
`coherence_floor`	0.1	0 to 1	The buzz-test floor of Section 2.5

Table 9. The settings of `band_average` for the frequency response update.

Parameter	Default	Range	What it changes
`use_tf_update`	off	on, off	Solve again from the controller's updated frequency response; the test's update mode must be on too
`estimate_frames`	20	1 to 100,000	Frames the controller's estimate must hold before an update is used
`drive_condition_number`	100	1 to 10^{12}	The largest condition number of a band's drive under the update
`min_drive_fraction`	0.02	0 to 1	Each drive's least share of the band's mean drive under the update
`hold_capped_bands`	off	on, off	Leave a band out of the update when its conditioning was capped
`min_coherence`	0	0 to 1	Leave a gauge's line out of the update below this multiple coherence; 0 is off
`tf_update_blend`	0	0 to 1	The weight of each update; 0 is a cumulative average
`n_e_seed`	0	0 to 10^{9}	Frames the identification counts for; 0 is its own count

Edge cases. A band whose lines carry no target gets no drive, and lines inside the band where no gauge has a target get none either. With more gauges than shakers the band's equations are solved by least squares; with fewer, the filtered minimum-norm solution. A rank-deficient band sensitivity is regularized by the threshold; under `regularization` none, singular values at or below the numerical floor are dropped. The first call returns the first solve; the gain correction starts at the second, with the drive the controller actually played (the law declares the input `LastDrive`). A gauge left out by the channel check stays in the solve with its gain multiplier reset to 1 until it comes back within the check.

What it gives up by design. It does not shape the response inside a band: inside a band with a resonance, a flat drive cannot follow the resonance (Figure 7), so line-by-line error stays where the band averages are met. The common price can leave a costly band short of its target. In exchange it does not chase line-by-line noise in the measured frequency response, and a small shift of a mode inside a band changes a band average less than it changes the lines. Decibel errors in fractional-octave bands are how multi-shaker tests are judged in parts of the published literature (Ross et al. 2017) and how the first article in this journal proposes a test be judged (Cross and Zwink 2026).

Source and differences. The method and its Python implementation are Brandon Zwink's; they are not published elsewhere. This section rests on the shipped port and its documentation at commit b587d5a1 (`band_average.rs`, whose law declaration names him as its author, the math crate and `docs/control-laws.md`); his Python law was not read for it. Hertzforge ships a Rust port of the Python law, held to it call for call by a parity

example in the math crate. The port leaves limit channels and the drive ceilings to the controller (its own RMS limit is off unless set), and differs from the Python law's inputs in two ways the code records: the last drive is the drive as the controller played it, and an update is accepted once the controller's exponentially averaged estimate holds `estimate_frames` frames (`band_average.rs`, module documentation). The shipped documentation records the first solve's sweep taking 0.5 to 3 s on a test of 48 gauges, 12 shakers and 8,193 lines, over the time a law is allowed on a 2 s loop (`docs/control-laws.md`). The L-curve, one of the threshold rules, is a known way of choosing a regularization amount, as Coletti and co-authors note (Coletti et al. 2026); Hansen's book on it (Hansen 2010) was not read, and the rule as built is the code's (`sweep.rs`, `l_curve_corner`).

3. The Protections Around Every Law

The law proposes and the controller disposes. Whatever law runs, the controller applies the same checks to its drive and the same protections to the run. Some of them change the drive without stopping anything (Table 10); some end the run (Table 11). A stop is recorded with which check made it, at which loop and level, and with the message listed after Table 11.

Table 10. The protections that change a law's drive without stopping the run (`guards.rs` of `crates/hertzforge-mimo-random`).

Protection	What it does	Default
Drive check	Refuses a drive of the wrong size, with a value that is not a number, not Hermitian (within 10^{-9}), or not zero outside the band. Repairs, and counts, a negative eigenvalue between 10^{-6} and 10^{-3} of the line's largest by clamping it and restoring the diagonal; refuses a larger one (`guards.rs`, `check_output`)	As stated
Change cap	Holds each drive's autospectrum to a largest rise and fall per loop, by a scaling that keeps every coherence and phase (`cap_change`)	+3.5 dB, −6 dB; settable to 6 dB and 20 dB
Drive ceiling	Scales each drive down to its RMS voltage ceiling (`apply_ceilings`)	Set per shaker
Limit channels	Scales a line's whole drive down where a limit channel's predicted response would pass its ceiling (`limit_responses`)	Set per channel

Table 11. The checks that end a run (`service.rs` and `config.rs` (`Protections`) of `crates/hertzforge-mimo-random`, and `LawGuards` in `crates/hertzforge-random/src/service/config.rs`). "Stop" is a controlled ramp of the drive down by 40 dB over 2 s by default; "abort" takes the drive to zero at once. While the soft start climbs, the alarm and abort lines are widened by 6 dB.

Check	What it judges	Default	Ends the run
Law guards	A law call that errors, fails, is late, or returns a refused drive is not used, and the drive in force is kept	A quarter of the loop's acquisition time; 3 in a row	Stop
Abort lines	The lines on which a gauge's autospectrum, from the averaged estimate, is outside its abort lines, once the estimate holds 16 degrees of freedom (`check_envelope`)	10 lines on one gauge	Abort
Growing error	Each gauge's RMS error over the band in decibels, from this loop's own estimate, against what the system can reach with more gauges than shakers and against the target otherwise (`errors_of`). A judged loop at or above the floor that does not improve on the gauge's best by 0.1 dB adds one to a count; a loop under the floor, or an improvement, resets it (`check_divergence`)	3 dB floor, 5 loops	Abort
Change cap held	For each drive, the lines on which the change cap held it back in at least 5 of the last 10 drives the law produced and the controller accepted (`check_cap_divergence`, `accept`)	10 lines of one drive	Abort
Drive voltage	A drive's measured RMS against its abort level	The ceiling plus 1 dB	Abort
Limit channel	A limit channel's measured spectrum against its ceiling and a margin	Set per channel	Abort
Kinematic limits, faults	A kinematic limit of an input channel, or a fault of the acquisition	Set per channel	Abort

The message each stop gives (`MimoTripKind::words` in `crates/hertzforge-mimo-random/src/report.rs`), with N, M, C, L, X and Y the numbers of the case:

- Law guards: “The control law did not produce a usable drive for 3 loops in a row (…); the run stopped.”
- Abort lines: “Control N was outside its abort lines on M lines (worst X dB).”
- Growing error: “Control N’s error grew for 5 loops in a row, to X dB.”
- Change cap held: “The change cap held drive N back on M lines for 5 loops in a row: the control law asked for more change than the core allows.”
- Drive voltage: “Drive N (channel C) reached X V rms against its Y V abort.”
- Limit channel: “Limit channel C was X dB over its ceiling on line L.”
- A kinematic abort names the channel, axis, value and limit; an acquisition fault gives the fault’s own description.

Two of the aborts judge the law rather than the article. The growing-error abort compares each gauge with what the system can reach (with more gauges than shakers, the least-squares response $PG_{ref}P^H$ from the identification) rather than with the target, so a law is not stopped for failing to reach what cannot be reached. It reads each loop’s own estimate, because an average over loops hides a drive that swings from loop to loop. The change-cap abort catches a law that swings the drive faster than the controller lets it, even when the averaged response looks right.

Two of the messages say more than the checks test. The growing-error check counts loops that do not improve on the gauge’s best by 0.1 dB; its message says the error “grew”. The change-cap check counts 5 of the last 10 accepted law drives; its message says “loops in a row”. The checks are what is described here; the messages are quoted as the code has them.

The defaults are protective and conservative on purpose. They exist to protect the shakers, the amplifiers and the article; a default that stops a good run costs time, and one that lets a bad run go on can cost hardware. Widening one for a real test is the test engineer’s decision for that test’s hardware. In particular, the growing-error count runs through the level steps, so a law still settling when it reaches full level can be stopped there. This article recommends no new default. A simulator has no hardware to protect, and the companion study states how it set these checks and why (Cross 2026b).

4. Plugging In a Law

A control law is a separate file loaded at run time: a WebAssembly component written against the interface `hertzforge:control-law@0.1.0`, world `random-mimo` (`wit/control-law/random-mimo.wit`). The six built-in laws are written once against the same boundary; the controller runs them natively and builds the same source into law files, and a test holds each law file to its native law within 10^{-12} relative (`crates/hertzforge-mimo-laws/tests/parity.rs`). The supported language is Rust, through Hertzforge’s law kit.

A law declares its parameters (each becomes an input box in the application, with its range and default), the system classes it supports, the optional inputs it needs and the diagnostics it reports. It implements two calls: `seed`, which returns the first drive from the identification, and `iterate`, which returns the next drive from each loop’s measurement. The worked example in the Hertzforge repository is the open-loop pseudo-inverse of Section 2.3 as a complete law crate to copy (`crates/hertzforge-law-sdk/template/mimo-open-loop`), built with `cargo build --target wasm32-wasip2 --release`.

The simulator runs any law. Before an imported law may drive a shaker, it must pass the controller’s check (`crates/hertzforge-random-laws/src/mimo_check.rs`): whole runs on six simulated articles (a modal article with four shakers and four control gauges, the same with measurement noise 30 dB under the smallest control gauge’s response at full level, a modal article with twelve control gauges and six shakers, the same run up a six-step level ladder, an ill-conditioned one and a rigid plate), under the controller’s own protections. Every case asks that each call was answered in time with a drive the checks accept and that no protection tripped

(the controller's own growing-error judgement is excused on the ill-conditioned article only). The four-shaker articles then ask every control gauge to come within 1 dB RMS of its target within 20 loops at full level; the twelve-gauge articles ask for a settled drive, each gauge's error changing by under 0.5 dB from loop to loop over three loops and within 6 dB; the ill-conditioned article asks only that the law survive. The pass is recorded against the law file's hash, so a changed file is checked again. A new law can also be run on the Open Benchmark Set (Cross 2026c), as the six are in the companion study (Cross 2026b).

5. How the Descriptions Were Checked Against the Code

Reading. Every statement about the build in Sections 1 to 4 was taken from the code at commit b587d5a1 and names the file and function it comes from. The law crates (`crates/hertzforge-mimo-laws`, `crates/hertzforge-law-band-average-math`) were read in full; the controller (`crates/hertzforge-mimo-random`) was read for the run order, the loop, the defaults and the protections. The settings tables were taken from each law's declared parameters, which are what the application shows. The law crates are identical at b587d5a1 and at the commit the companion study ran, 646c4ca4, which adds only the study's runner and player.

A numerical check. Each law's equations, as this article writes them, were implemented again in Python without Hertzforge (`code/paper_laws.py`) and compared with the shipped laws, run unchanged on the same made-up structures by `code/shipped_laws` (Rust, built against the code at 646c4ca4). `code/check_against_shipped.py` makes the comparison and prints it (`data/check_against_shipped_output.txt`). The structures are small modal models (Appendix B gives one line of one), and every measurement in the check is exact: the response is $HG_{uu}H^H$ of the structure, with no noise and no averaging, and no protection of Section 3 is applied. Table 12 lists what was compared.

Table 12. The numerical check of the article's equations against the shipped laws. Values in decibels are written by the shipped program to ten significant digits, which sets the floor near 1e-10 dB.

Law	What was compared	On what	Largest difference
`pinv_open`	Each gauge's RMS error over the band, every loop	2 gauges, 2 shakers, 1,161 lines, 31 loops, measured frequency response 3 percent off	3.8e-10 dB
`pinv_closed`	Each gauge's error on every line after 0 and 5 loops	The same	5.0e-10 dB
`pinv_closed`	Each gauge's RMS error over the band, loops 0 to 22	The same	3.8e-10 dB
`pinv_closed`	The same, loops 23 to 30	The same	1.1e-3 dB; one line, 0.53 dB (see below)
`shape_svd`	Directions kept on each line	6 gauges, 4 shakers, 1,161 lines	None differ
`buzz_match`	The target's coherence and phase of one pair	2 gauges, 3 shakers, 1,161 lines	5.0e-11 in coherence, 4.8e-8 degrees
`min_drive`	Drive power and the target's autospectra at every point of the trade-off grid, and the automatic choice	One line of a 2 by 2 structure, 13 trade-offs	9.1e-8 of the largest drive power; 2.4e-7 in a target ratio; same choice
`band_average`	The first solve's drive and each band's threshold	1 gauge, 1 shaker, 921 lines in 40 to 500 Hz	3.8e-10 relative in the drive; no threshold differs

Wherever the comparison is not limited by rounding, the largest difference is 5.0e-10 dB, at the precision the shipped program writes. The exception is the closed loop from loop 23. On 12 lines near 75, 87, 155 and 404 Hz, the paper's implementation and the shipped law part from loop 23: by 1.1e-3 dB in the RMS over the band and by up to 0.53 dB on one line after 30 loops. The paper's own implementation, run again with the measured frequency response changed by one or two parts in 10^{15}, parts from itself by the same amounts (1.3e-3 dB in the RMS, 0.50 dB on one line). On those lines gauge 2's autospectrum has reached its target to within rounding while its cross term is still held at zero by the second form. There the sign of E_{kk}, which picks the second form or the first (`factor.rs`, `FactorLoop::iterate`, the test `ekk < 0.0`, with no dead band), is set by rounding, and the first form, when picked, moves the cross term by a finite step. A last-bit difference can therefore flip that one choice; the hold, the limits, the bound, the floor and the halving do not act on those lines. The difference after loop 22 is therefore of the size rounding alone produces, and is not a difference between the equations and the build.

What the check does not cover. It covers each law's arithmetic as described, not the controller. The closed loop was checked on a system with as many shakers as gauges, in the gauges' own coordinates, with truncation and no weighting; the reduced coordinates, Tikhonov damping, weighting and the carrying over of the factor when the frequency response changes were read but not checked numerically. The shape-constrained law was checked on its choice of directions; its loop is the shared loop. The buzz-test target was checked on one pair, where no repair is needed. The minimum-drive law was checked on one line of a square system, solved by a different method (Appendix A.8) from the build's dual barrier, so the agreement checks the result and not the steps. The band-average law was checked on its first solve with one gauge and one shaker; the gain correction, the constrained re-fit, cross terms and the frequency response update were read but not checked numerically.

Two independent checks. An agent that did not write the article checked each statement about the build against the code at this commit, wrote its own closed loop from `factor.rs`, ran the numerical check, and recomputed every number of Appendix B. Its own closed loop matched the shipped law to 4e-11 dB through loop 22 and parted from it on the same 12 lines by the same amounts. A second check, of the text changed after the first, ported the correction in Smallwood's appendix line by line and ran it on the case of Figure 3. It confirmed every number the article quotes from its own code, and found three statements to correct, seven to tighten and two omissions. The changes both checks asked for were made in this version, and the second check's port is in the article's code (`code/smallwood_port`).

6. Limits

What this article does not claim. It reports no score and no ranking, and says nothing about how well any law does on any test. Every figure and every number is from small made-up structures, chosen to show what a law does. Everything is code and simulation; nothing here has run on hardware. Where a law's design intent is stated, it is intent, not a finding. The companion study (Cross 2026b) reports how each law scores on the Open Benchmark Set.

A gauge held high near a mode. Where the measured frequency response is off near a mode, the drive leaks into a gauge, the gauge reads high, and the cross term that could cancel the leak would have to grow. The second form forbids a cross term to grow while its gauge is high, so the loop cuts that gauge's diagonal term to the floor and the gauge stays high by the leak. In the build this holds gauge 2 near 90 Hz in Figure 3 (Appendix B.8): +4.87 dB at 91 Hz at loop 30, with the independent target, a banded one, and a target with cross terms (coherence 0.64, phase 30 degrees) alike. Smallwood's second form, as published, can hold a gauge high in the same way. A line-by-line port of his Appendix A (`code/smallwood_port`, its output in `data/smallwood_port_output.txt`), run on the same case at the same convergence factor of 0.5, holds 90.5 to 93 Hz at the build's levels when the target has cross terms, and holds 91.5 and 92 Hz (+5.04 and +4.61 dB) with the independent target. With the independent

target it releases the line at 91 Hz (+0.40 dB at loop 30) and brings gauge 2′s RMS error from 85 to 96 Hz to 1.54 dB, against the build's 2.37 dB. Any one of four of the build's departures, put alone into the port, holds the 91 Hz line again: the second form on a cross term at zero (+4.87 dB, and 2.37 dB from 85 to 96 Hz, the build's figures), the second-form step without the index (+4.86 dB), the order of the two forms (+4.87 dB), or the limit on the diagonal (+4.77 dB); the build's divisor alone does not (+0.40 dB). The release also depends on how the first-column line of his listing is read (as printed, +0.40 dB; read like the other columns, +4.86 dB) and on the convergence factor (at 1, his own example's value, and at 0.25 the port holds the line, at +4.87 and +4.80 dB).

What the build gains from its departures on this case is as plain. Near 160 Hz the port does not hold gauge 2 steady: at 161 Hz it swings from −5.71 dB (loop 3) to +2.52 dB (loop 15) and ends at −4.23 dB, where the build holds +2.97 dB from loop 3 on. With the target that has cross terms and a convergence factor of 1, the port diverges near 413 Hz (+132.7 dB at loop 30, with 103 of its own warnings of a diagonal correction above 10^6); the build does not. Near 160 Hz on the independent target the hold is the build's own departure, the second form on a term at zero: gauge 2′s RMS error from 150 to 172 Hz is 1.12 dB with the independent target and 0.40 dB with the target that has cross terms. These are behaviors on one made-up case, reported as the port gave them; they are no general statement about Smallwood's method or the build. Reproduction: `code/shipped_laws`, then `code/fig3_closed_loop.py`; `code/zero_term_trace.py`; `code/smallwood_port/run_fig3_case.py`. How often either hold matters on a real test is not known.

Two of Smallwood's rules added to the build's loop. The article's code adds two of his rules against a term at zero (the second form only for a term that is not zero; a term at zero set to the floor) to its own implementation of the build's loop, as an option, without his other differences (Table 2) (`code/paper_laws.py`, `loop_step`; `code/zero_term_trace.py`). The experiment is in the article's code, not in Hertzforge, and it does not show what his algorithm does; the port above does. On the structure of Figure 3 with the independent target, the rules release the lines near 160 Hz (gauge 2′s RMS error from 150 to 172 Hz at loop 30, 1.12 dB without them and 0.52 dB with them; over the band, 0.392 and 0.322 dB) but not those near 90 Hz (2.37 and 2.22 dB), and the released term overshoots: at loop 5 the line at 91 Hz reads 17.6 dB low, where without them no line is more than 7.2 dB off at that loop. With a banded specification (levels stepping at 150 and 300 Hz) they make gauge 2 worse, 0.923 to 0.983 dB over the band and 4.45 to 5.37 dB near 90 Hz, though better near 160 Hz (2.79 to 2.25 dB); with the target that has cross terms nothing changes. His whole algorithm, ported, ends the 91 Hz line on the independent target at +0.40 dB where this graft ends it at +4.74 dB. On this experiment's numbers the shipped loop does not take the two rules. These numbers describe the loop on one made-up structure, not how any law scores.

What is not built. The leak comes from the error in the measured frequency response; with an exact one there is none. The controller can measure the frequency response again during the run, blended with the identification's by a weight the test sets (`FrfMode::Update` in `config.rs`, Section 1.3); nothing here tests whether that releases these lines. A second form that may grow a cross term on a gauge above its target is not built.

Rounding. As Section 5 shows, on lines where a gauge has reached its target to within rounding while its cross term is held at zero, the loop's later result depends on the last bits of its inputs. The native law and its law file are held to each other within 10^{-12} relative by a test (Section 4); runs that go on longer, on lines like these, can differ between the two by more.

The minimum-drive floor. The largest shortfall (3 dB by default) bounds each gauge's average shortfall over the sampled lines, not each line, and only the alarm lines bound single lines.

Messages and help texts. Two protections' messages say more than their checks test (Section 3). Some of the software's own help texts and messages describe these behaviors differently from the code at this commit; the article follows the code.

Departures from the sources. Each law's section lists where the build differs from its published source. Six sources were not read or not obtained: Daborn's thesis (Daborn 2014), credited as Schultz credits it (Schultz 2019); Smallwood's minimum input trace paper (Smallwood 2013) and Musella and co-authors (Musella et al. 2018), credited as Coletti and co-authors describe them (Coletti et al. 2026); Dodds and Robson (Dodds and Robson 1975), credited as Smallwood credits them (Smallwood 1999); Hansen's book (Hansen 2010); and the journal version of the shape-constrained method (Schultz and Avitabile 2020). This article makes no claim about their content beyond what the sources that cite them say.

One build. The descriptions are of main at b587d5a1. A later change to a law changes what is described here, and the article's code (Section 5) is the way to check a later build against it.

7. Appendices

Appendix A. Derivations

A.1 The forward relation. On one line, the response at the gauges is $y = Hu$, with u the drives. Then $G_{yy} = E[yy^H] = H\,E[uu^H]\,H^H = HG_{uu}H^H$, which is Equation 1.

A.2 The pseudo-inverse drive. Let $H = U_r S_r V_r^H$ be the thin decomposition over its r non-zero singular values, and $H^+ = V_r S_r^{-1} U_r^H$. *More gauges than shakers.* Any response HXH^H lies in $U_r \mathcal{M} U_r^H$, with $\mathcal{M} = S_r V_r^H X V_r S_r$ an r by r matrix that X can make any positive semi-definite matrix. Write G_{ref} in the basis $[U_r, U_\perp]$; the squared Frobenius distance $\left\|G_{ref} - U_r \mathcal{M} U_r^H\right\|^2$ is $\left\|U_r^H G_{ref} U_r - \mathcal{M}\right\|^2$ plus terms that do not depend on $\mathcal{M}$, so the least distance over all Hermitian $\mathcal{M}$ is at $\mathcal{M} = U_r^H G_{ref} U_r$, which is positive semi-definite and so reachable, that is $X = V_r S_r^{-1} U_r^H G_{ref} U_r S_r^{-1} V_r^H = H^+ G_{ref} H^{+H}$, and the response is $U_r U_r^H G_{ref} U_r U_r^H = P G_{ref} P^H$ with $P = HH^+ = U_r U_r^H$. The drive is positive semi-definite when G_{ref} is. *Fewer gauges than shakers, full rank.* Write a drive in the basis $[V_r, V_\perp]$, with blocks X_{rr}, $X_{r\perp}$, $X_{\perp\perp}$. Since $HV_\perp = 0$, the response is $U_r S_r X_{rr} S_r U_r^H$, which equals G_{ref} only when $X_{rr} = S_r^{-1} U_r^H G_{ref} U_r S_r^{-1}$. The total power is $\mathrm{tr}\ X = \mathrm{tr}\ X_{rr} + \mathrm{tr}\ X_{\perp\perp} \geq \mathrm{tr}\ X_{rr}$, with equality when $X_{\perp\perp} = 0$, which for a positive semi-definite X forces $X_{r\perp} = 0$. The least-power drive is therefore $V_r X_{rr} V_r^H = H^+ G_{ref} H^{+H}$.

A.3 Every drive of the loop can be played. With $B = AZ$, $G_{uu} = BB^H$, and for any vector x, $x^H BB^H x = \left\|B^H x\right\|^2 \geq 0$.

A.4 The first and second forms. $(Z + \Delta)(Z + \Delta)^H = ZZ^H + \Delta Z^H + Z\Delta^H + \Delta\Delta^H$. Setting this equal to G_{ref} with $ZZ^H = G_{yy}$ (a measured frequency response that is exact, in the loop's coordinates) and dropping the second-order term gives Equation 5. With Z and Δ lower triangular, entry (k, j), $k \geq j$, of $\Delta Z^H + Z\Delta^H$ is $\sum_{m \leq j} (\Delta_{km} \overline{Z_{jm}} + Z_{km} \overline{\Delta_{jm}})$, which involves column j of Δ and columns before it only. For $k = j$ with Z_{jj} and Δ_{jj} real, the $m = j$ term is $2Z_{jj}\Delta_{jj}$; for $k > j$ it is $\Delta_{kj} Z_{jj} + Z_{kj} \Delta_{jj}$. Solving each for the unknown of column j gives Equation 6. The second form is a rule, not a derived step. If every term of row k of Z were scaled by $1 + \delta$, $(ZZ^H)_{kk}$ would scale by $(1 + \delta)^2 \approx 1 + 2\delta$, and $\delta = E_{kk} / (2G_{ref,kk})$ would remove the relative excess $E_{kk}/G_{ref,kk}$ (negative when the gauge is over) to first order. Changing one term alone changes $(ZZ^H)_{kk}$ by about $2\delta |Z_{kj}|^2$ instead. The build takes $\Delta_{kj} = \delta Z_{kj}$ for each cross term the second form governs (the first case of Equation 7); the row's diagonal takes the first form of its own column, whose sum already subtracts the change in the cross terms. When the gauge is half again over its target or more, the build halves the term instead (`factor.rs`, `FactorLoop::iterate`). Smallwood's Equation 13 has twice the row index times the column gauge's target in the denominator and the column gauge's error in the numerator; his appendix follows it in the first column and uses the column index and the row gauge's error and target in the others (Smallwood 1999). The build divides by $2\,G_{ref,kk}$, with the row gauge's error, in every column (Table 2).

A.5 The reduced coordinates. Without weighting and with truncation, $A = V_q S_q^{-1}$ and $T = U_q^H$, so $HA = U_q$ on a system whose measured frequency response is exact. The response is $HAZZ^HA^HH^H = U_qZZ^HU_q^H$ and its reduced form is $TG_{yy}T^H = ZZ^H$: the loop in these coordinates is square and full rank, and drives ZZ^H to the reduced target $U_q^HG_{ref}U_q$. The response then settles at $U_qU_q^HG_{ref}U_qU_q^H = PG_{ref}P^H$. The seed, Z the factor of the reduced target, gives $G_{uu} = V_qS_q^{-1}U_q^HG_{ref}U_qS_q^{-1}V_q^H = H^+G_{ref}H^{+H}$, the drive of Equation 2. With weighting, H is replaced by WH and $T = U_q^HW$ (the code's documentation states $AT = H^+$).

A.6 The shape constraint. With $C = V_m$, $HC = USV^HV_m = U_mS_m$, whose pseudo-inverse is $S_m^{-1}U_m^H$. Then $C(HC)^+ = V_mS_m^{-1}U_m^H$, and $G_{uu} = CG_cC^H = V_mS_m^{-1}U_m^HG_{ref}U_mS_m^{-1}V_m^H$, the pseudo-inverse drive truncated to m directions.

A.7 The buzz-test target. B is a spectral density matrix, so positive semi-definite. With $\Lambda = \mathrm{diag}(1/\sqrt{B_{jj}})$, $C = \Lambda B\Lambda$, a congruence, is positive semi-definite, and so is DCD. Setting entries of C to zero is not a congruence, so it can produce a negative eigenvalue; hence the repair.

A.8 The minimum-drive problem. *Its dual.* With $w_k|r_k| = \max_{|\nu_k|\le w_k} \nu_k r_k$, Equation 9 is

$$\min_{X\succcurlyeq 0} \max_{|\nu_k|\le w_k} \sum_k \nu_k(y_k - h_kXh_k^H) + \lambda \ \mathrm{tr}\ X = \min_{X\succcurlyeq 0} \max_{\nu} [\nu^T y + \mathrm{tr}\ (XS(\nu))]$$

with $S(\nu) = \lambda I - H^H\ \mathrm{diag}(\nu)H$. The problem is convex in X and linear in ν over a compact set, so the minimum and maximum exchange; $\min_{X\succcurlyeq 0} \mathrm{tr}(XS)$ is 0 when $S \succcurlyeq 0$ and unbounded below otherwise, which gives the dual of Section 2.6. Adding $\mu[\log\det S + \sum_k \log(w_k - \nu_k) + \sum_k \log(w_k + \nu_k)]$ to the dual objective and setting its derivative to zero gives $y_k = \mu(HS^{-1}H^H)_{kk} + \mu/(w_k - \nu_k) - \mu/(w_k + \nu_k)$, so $X = \mu S^{-1}$ has residuals $y_k - (HXH^H)_{kk} = 2\mu\nu_k/(w_k^2 - \nu_k^2)$, which vanish with μ unless $|\nu_k| \to w_k$ (`pareto.rs`, module documentation). *Two gauges, two shakers, the check's method.* On a line where H is square and invertible, write the target $Y = HXH^H$; then $\mathrm{tr}\ X = \mathrm{tr}(FY)$ with $F = H^{-H}H^{-1}$. With the target's autospectra d_1, d_2 held and cross term c, $|c|^2 \le d_1d_2$, $\mathrm{tr}(FY) = F_{11}d_1 + F_{22}d_2 + 2\ \mathrm{Re}(F_{12}\bar{c})$, least at $c = -\sqrt{d_1d_2}\,F_{12}/|F_{12}|$: full coherence, at one phase. So the least drive power for autospectra d is $F_{11}d_1 + F_{22}d_2 - 2|F_{12}|\sqrt{d_1d_2}$, and Equation 9 becomes a problem in two non-negative unknowns, which `code/paper_laws.py` (`min_drive_line`) solves directly. It is the curve of Figure 6 (left).

Appendix B. A Worked Example

One line, 150 Hz, of the two-gauge, two-shaker made-up structure of Figure 3, carried through the shared loop and each law. Every number below is printed by `code/worked_example.py` (`data/worked_example_output.txt`), which uses only `code/paper_laws.py`. The target is independent, with autospectra 1 and 0.5 (made-up units, g²/Hz). The measured frequency response is that of the same structure with every mode 3 percent high, as if the structure had moved between identification and test.

B.1 The line. Table 13 gives the structure on this line and its measured frequency response (g/V).

Table 13. The structure and its measured frequency response at 150 Hz.

	Shaker 1	Shaker 2
Structure, gauge 1	−0.4266 - 0.0158i	0.9147 - 0.3731i
Structure, gauge 2	−0.9198 + 0.0648i	−5.0781 + 1.4474i
Measured, gauge 1	−0.4272 - 0.0080i	0.5569 - 0.1808i
Measured, gauge 2	−0.8456 + 0.0225i	−3.6657 + 0.6834i

The structure's singular values are 5.4364 and 0.5794 (condition number 9.38); the measured frequency response's are 3.8540 and 0.5404 (condition number 7.13). Both singular values are far above 10^{-3} of the largest, so truncation keeps both, and the loop runs in the gauges' own coordinates.

B.2 Pseudo-inverse, open loop. Equation 2 gives drive autospectra 3.2445 and 0.1860 V²/Hz with drive coherence 0.8090. On the measured frequency response this drive gives exactly 1 and 0.5. On the structure it gives 1.3402 and 1.1514, errors of +1.27 and +3.62 dB, with a response coherence of 0.26 where the target asks for none: the error in the measured frequency response passes straight to the test.

B.3 Pseudo-inverse, closed loop. The target's factor is $R = \mathrm{diag}(1, 0.7071)$ and Z starts there. Table 14 follows the first eight loops.

Table 14. The shared loop on one line of the worked example, at the default settings: each gauge's error in dB, the factor's terms Z_{11}, Z_{21} and Z_{22}, and the drive power in V²/Hz.

Loop	Gauge 1 (dB)	Gauge 2 (dB)	Diagonal 1	Cross term	Diagonal 2	Drive power
0	+1.272	+3.622	1.0000	0	0.7071	3.4305
1	+0.494	+0.935	0.9150	0	0.4768	2.8487
2	+0.175	+0.037	0.8820	0	0.4138	2.6423
3	+0.059	−0.041	0.8704	0	0.4112	2.5734
4	−0.151	−2.003	0.8664	0.2569 - 0.0551i	0.4141	2.7374
5	−0.087	−0.617	0.8763	0.3119 - 0.0581i	0.4911	2.8494
6	−0.045	−0.264	0.8820	0.3331 - 0.0566i	0.5115	2.9042
8	−0.010	−0.051	0.8862	0.3459 - 0.0552i	0.5239	2.9431

For three loops gauge 2 is above its target, so the second form holds the cross term Z_{21} at zero and only the diagonal terms move. At loop 3 gauge 2 dips just under its target, $E_{22} > 0$, the first form applies, and the cross term grows to cancel the leak; the first step overshoots gauge 2 by 2 dB, and the loop then converges. On a line where gauge 2 never dips below its target the term is never released (B.8, Section 6).

B.4 Shape-constrained. The ratio of the singular values is $0.5404/3.8540 = 0.1402$, under the default 0.2, so one direction is kept. The drive power falls from 3.4305 to 0.0342 V²/Hz, and the response is 0.0267 and 0.9815 (-15.73 and +2.93 dB): on this line, between modes, one direction cannot reach both gauges, which is the trade the law makes by design.

B.5 Buzz-test matched. Under uncorrelated unit drives, the identification response gives the pair a coherence of 0.4352 at 170.60 degrees, above the floor of 0.1. The modified target keeps the autospectra 1 and 0.5 and takes the cross term $\sqrt{1 \times 0.5} \times C_{12} = -0.4602 + 0.0762i$. Its first drive needs 3.0373 V²/Hz against 3.4305 for the independent target, with a drive coherence of 0.91.

B.6 Minimum drive, this line alone. With relative weights, $r_0 = 2$ and $d_0 = 1.9376$ V²/Hz. For trade-offs 10^{-6} to 10^{-1} the target meets both autospectra exactly with a drive power of 2.8345 V²/Hz, the least that meets them (against 3.4305 for the independent target); at 1 the drive falls to 0.0385 and gauge 1 keeps 1.5 percent of its specification; from 100 up there is no drive. Without alarm lines the automatic rule leaves out the points above nine tenths of the largest drive power or error, which removes the exact-control points and the no-drive

points, and the largest shortfall removes those that give up gauge 1; nothing is left, and the smallest trade-off, 10^{-6}, is taken.

B.7 Band average. The made-up one-gauge, one-shaker structure of Figure 7, band 141.254 to 158.489 Hz (34 lines, the one-sixth-octave band that holds 150 Hz). The band target is 1, the band sensitivity 0.10752, and the sweep's common price gives this band the largest threshold, 10^{5}. The flat drive is 9.3007 and the band response 1.0000 (0.000 dB); at 150 Hz itself the response is 0.9972 (-0.01 dB). This band holds no resonance, so its lines sit close to the band average; in the band near 120 Hz they do not (Figure 7).

B.8 The line at 91 Hz. The same structure and target, the line at 91 Hz, near the first mode (90 Hz in the structure, 92.7 Hz in the measured frequency response, whose condition number there is 49.8). Every number is printed by `code/zero_term_trace.py` (`data/zero_term_trace_output.txt`). The loop's gain on this line, the structure times the inverse the loop uses, has eigenvalues 1.091 and 0.394 - 1.196i: a correction reaches the gauges turned by up to 72 degrees. The leak from source 1 into gauge 2 has a power gain of 2.77. Table 15 follows the factor's terms (Z_{11}, $|Z_{21}|$, Z_{22}).

Table 15. The shared loop on the line at 91 Hz of the worked example: the factor's terms and gauge 2's error.

Loop	Diagonal 1	Cross term	Diagonal 2	Gauge 2 (dB)
0	1.000	0	1	+5.67
1	1.003	0	0.501	+4.80
2	1.020	0	0.251	+4.69
3	1.033	0	0.126	+4.74
5	1.046	0	0.0316	+4.82
10	1.051	0	0.001	+4.86
15	1.051	0	0.0001	+4.87
30	1.051	0	0.0001	+4.87

Gauge 2 is above its target from the first loop. The first form would grow Z_{21}; the second form forbids it, and with Z_{21} at zero it does not move. Gauge 2's own diagonal correction is negative every loop, the per-loop limit cuts Z_{22} by half each loop, and it reaches the floor, 10^{-4}, by loop 15. Gauge 2 then reads the leak from source 1 alone, $2.77 \times 1.051^{2} = 3.07$, 4.87 dB over its target. Cancelling the leak would take a cross term of magnitude about 1.83. With a target that has cross terms (coherence 0.64, phase 30 degrees) the line ends at the same 4.87 dB. With two of Smallwood's rules added to the build's loop (Section 6) the cross term is released at the first loop, grows to 1.48 by loop 5 while the turned correction overshoots (gauge 2 at −17.61 dB), and is then shrunk by the second form; the line ends at +4.74 dB at loop 30. A port of Smallwood's whole Appendix A releases this line on the independent target (+0.40 dB at loop 30, from +5.67, +2.04, +1.32, +0.64 and +0.38 dB at loops 0 to 4); with the target that has cross terms it holds it at +4.87 dB, as the build does (`code/smallwood_port`, Section 6).

Contribution Statement

Kevin Cross is the sole author. He chose the five laws from the literature, directed their implementation in Hertzforge, and directed this description of them and its check. The band-average law described in Section 2.7 is Brandon Zwink's: he designed it and wrote its Python implementation, which the shipped law is ported from.

How This Was Made

The text, the figures, the code that draws them and the code of the numerical check were written with Claude Code, an AI coding assistant running Anthropic's Claude Opus models, under Kevin Cross's direction. The author is responsible for every word, number and line of code.

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References

Coletti, Keaton, R. Benjamin Davis, and Ryan Schultz. 2026. "Pareto-Optimal Target Definition for Multi-Axis Random Vibration Testing." *Mechanical Systems and Signal Processing* 249: 114075. https://doi.org/10.1016/j.ymssp.2026.114075.

Cross, Kevin. 2026a. "Multi-Shaker Random Vibration Control: What Is Published, and Where It Stops." *Hertzforge Journal* 1: 2026–05. https://doi.org/10.5281/zenodo.24254415.

Cross, Kevin. 2026b. *Six Control Laws for Multi-Shaker Random Vibration: How Each Scores on the Open Benchmark Set*. Hertzforge Journal, in preparation.

Cross, Kevin. 2026c. *The Open Benchmark Set: Public Test Problems, a Fair Run and a Scorecard for Multi-Shaker Control Laws*. Hertzforge Journal, in preparation.

Cross, Kevin, and Brandon Zwink. 2026. "What Is a Good Multi-Shaker Test? A Proposed Practice for Judging MIMO Vibration and Shock Tests on Flexible Articles." *Hertzforge Journal* 1: 2026–01. https://doi.org/10.5281/zenodo.24252422.

Daborn, P. M. 2014. "Smarter Dynamic Testing of Critical Structures." PhD thesis, University of Bristol.

Dodds, C. J., and J. D. Robson. 1975. "Partial Coherence in Multivariate Random Processes." *Journal of Sound and Vibration* 42 (2): 243–49. https://doi.org/10.1016/0022-460X(75)90219-9.

Hansen, Per Christian. 2010. *Discrete Inverse Problems: Insight and Algorithms*. SIAM. https://doi.org/10.1137/1.9780898718836.

Hunter, Norman F., Kevin R. Cross, and Garrett Nelson. 2018. "The Cross Spectrum in Multiple Input Multiple Response Vibration Testing." *Topics in Modal Analysis and Testing, Volume 9, Proceedings of IMAC XXXVI*, 91–102. https://doi.org/10.1007/978-3-319-74700-2_10.

Musella, Umberto, Giacomo D'Elia, Alex Carrella, et al. 2018. "A Minimum Drives Automatic Target Definition Procedure for Multi-Axis Random Control Testing." *Mechanical Systems and Signal Processing* 107: 452–68. https://doi.org/10.1016/j.ymssp.2018.01.039.

Ross, Michael, Laura D. Jacobs, Greg Tipton, et al. 2017. “6-DOF Shaker Test Input Derivation from Field Test.” *Shock and Vibration, Aircraft/Aerospace, Energy Harvesting, Acoustics and Optics, Volume 9, Proceedings of IMAC XXXV*, 11–22. https://doi.org/10.1007/978-3-319-54735-0_2.

Schultz, R., and P. Avitabile. 2019. *Shape-Constrained Input Estimation for Multi-Shaker Vibration Testing*. Conference paper proposed for IMAC XXXVIII, 2020; SAND2019-12631C, OSTI 1642864. https://www.osti.gov/biblio/1642864.

Schultz, R., and P. Avitabile. 2020. “Shape-Constrained Input Estimation for Efficient Multi-Shaker Vibration Testing.” *Experimental Techniques* 44 (4): 409–23. https://doi.org/10.1007/s40799-020-00361-0.

Schultz, Ryan A. 2019. “A Demonstration of Force Estimation and Regularization Methods for Multi-Shaker Testing.” *Sensors and Instrumentation, Aircraft/Aerospace, Energy Harvesting and Dynamic Environments Testing, Volume 7, Proceedings of IMAC XXXVII*, 229–43. https://doi.org/10.1007/978-3-030-12676-6_21.

Smallwood, D. O. 1999. “Multiple Shaker Random Vibration Control: An Update.” *Institute of Environmental Sciences and Technology Annual Meeting*. https://www.osti.gov/biblio/3874.

Smallwood, D. O. 2013. “Minimum Input Trace for Multiple Input Multiple Output Linear Systems.” *Journal of the IEST* 56 (2): 57–67. https://doi.org/10.17764/jiet.56.2.41344333010617q6.

Smallwood, David O. 2007. “Multiple-Input Multiple-Output (MIMO) Linear Systems Extreme Inputs/Outputs.” *Shock and Vibration* 14 (2): 107–31. https://doi.org/10.1155/2007/701837.

Implemented In

The six laws, the control service and the simulator are in Hertzforge, github.com/Hertzforge/Hertzforge, main branch at commit b587d5a1: `crates/hertzforge-mimo-laws`, `crates/hertzforge-law-band-average-math` and `crates/hertzforge-mimo-random`. The numerical check and the worked example are in this article’s `code/`.

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